Windmills have long been used to pump water from wells, grind grain, and saw wood. They are more recently being used to produce electricity. The propeller radius of these windmills range from one to one hundred meters, and the power output ranges from a hundred watts to a thousand kilowatts. See page 800 in Section 10.1 for some more information and a question and answer about windmills.
Chapter 10 Overview

By the beginning of the 17th century, algebra and geometry had developed to the point where physical behavior could be modeled both algebraically and graphically, each type of representation providing deeper insights into the other. New discoveries about the solar system had opened up fascinating questions about gravity and its effects on planetary motion, so that finding the mathematical key to studying motion became the scientific quest of the day. The analytic geometry of René Descartes (1596–1650) put the final pieces into place, setting the stage for Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716) to stand “on the shoulders of giants” and see beyond the algebraic boundaries that had limited their predecessors. With geometry showing them the way, they created the new form of algebra that would come to be known as the calculus.

In this chapter we will look at the two central problems of motion much as Newton and Leibniz did, connecting them to geometric problems involving tangent lines and areas. We will see how the obvious geometric solutions to both problems led to algebraic dilemmas, and how the algebraic dilemmas led to the discovery of calculus. The language of limits, which we have used in this book to describe asymptotes, end behavior, and continuity, will serve us well as we make this transition.
**Instantaneous Velocity**

Galileo experimented with gravity by rolling a ball down an inclined plane and recording its approximate velocity as a function of elapsed time. Here is what he might have asked himself when he began his experiments:

**A Velocity Question**

A ball rolls a distance of 16 feet in 4 seconds. What is the instantaneous velocity of the ball at a moment of time 3 seconds after it starts to roll?

You might want to visualize the ball being frozen at that moment, and then try to determine its velocity. Well, then the ball would have zero velocity, because it is frozen! This approach seems foolish, since, of course, the ball is moving.

Is this a trick question? On the contrary, it is actually quite profound—it is exactly the question that Galileo (among many others) was trying to answer. Notice how easy it is to find the *average* velocity:

\[
\text{\textit{v}}_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{16 \text{ feet}}{4 \text{ seconds}} = 4 \text{ feet per second.}
\]

Now, notice how inadequate our algebra becomes when we try to apply the same formula to *instantaneous* velocity:

\[
\text{\textit{v}}_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{0 \text{ feet}}{0 \text{ seconds}},
\]

which involves division by 0—and is therefore undefined!

So Galileo did the best he could by making \(\Delta t\) as small as experimentally possible, measuring the small values of \(\Delta s\), and then finding the quotients. It only *approximated* the instantaneous velocity, but finding the exact value appeared to be algebraically out of the question, since division by zero was impossible.

**Limits Revisited**

Newton invented “fluxions” and Leibniz invented “differentials” to explain instantaneous rates of change without resorting to zero denominators. Both involved mysterious quantities that could be infinitesimally small without really being zero. (Their 17th-century colleague Bishop Berkeley called them “ghosts of departed quantities” and dismissed them as nonsense.) Though not well understood by many, the strange quantities modeled the behavior of moving bodies so effectively that most scientists were willing to accept them on faith until a better explanation could be developed. That development, which took about a hundred years, led to our modern understanding of limits.

Since you are already familiar with limit notation, we can show you how this works with a simple example.
EXAMPLE 2  Using Limits to Avoid Zero Division

A ball rolls down a ramp so that its distance $s$ from the top of the ramp after $t$ seconds is exactly $t^2$ feet. What is its instantaneous velocity after 3 seconds?

SOLUTION  We might try to answer this question by computing average velocity over smaller and smaller time intervals.

On the interval $[3, 3.1]$: 

$$\frac{\Delta s}{\Delta t} = \frac{(3.1)^2 - 3^2}{3.1 - 3} = \frac{0.61}{0.1} = 6.1 \text{ feet per second.}$$

On the interval $[3, 3.05]$: 

$$\frac{\Delta s}{\Delta t} = \frac{(3.05)^2 - 3^2}{3.05 - 3} = \frac{0.3025}{0.05} = 6.05 \text{ feet per second.}$$

Continuing this process we would eventually conclude that the instantaneous velocity must be 6 feet per second.

However, we can see directly what is happening to the quotient by treating it as a limit of the average velocity on the interval $[3, t]$ as $t$ approaches 3:

$$\lim_{{t \to 3}} \frac{\Delta s}{\Delta t} = \lim_{{t \to 3}} \frac{t^2 - 3^2}{t - 3}$$

$$= \lim_{{t \to 3}} \frac{(t + 3)(t - 3)}{t - 3}$$

$$= \lim_{{t \to 3}} (t + 3) \cdot \frac{t - 3}{t - 3}$$

$$= \lim_{{t \to 3}} (t + 3)$$

$$= 6$$

Notice that $t$ is not equal to 3 but is approaching 3 as a limit, which allows us to make the crucial cancellation in the second to last line of Example 2. If $t$ were actually equal to 3, the algebra above would lead to the incorrect conclusion that $0/0 = 6$. The difference between equaling 3 and approaching 3 as a limit is a subtle one, but it makes all the difference algebraically.

It is not easy to formulate a rigorous algebraic definition of a limit (which is why Newton and Leibniz never really did). We have used an intuitive approach to limits so far in this book and will continue to do so, deferring the rigorous definitions to your calculus course. For now, we will use the following informal definition.

DEFINITION (INFORMAL)  Limit at $a$

When we write “$\lim \ f(x) = L$,“ we mean that $f(x)$ gets arbitrarily close to $L$ as $x$ gets arbitrarily close (but not equal) to $a$. 

ARBITRARILY CLOSE

This definition is useless for mathematical proofs until one defines “arbitrarily close,” but if you have a sense of how it applies to the solution in Example 2 above, then you are ready to use limits to study motion problems.
The Connection to Tangent Lines

What Galileo discovered by rolling balls down ramps was that the distance traveled was proportional to the square of the elapsed time. For simplicity, let us suppose that the ramp was tilted just enough so that the relation between $s$, the distance from the top of the ramp, and $t$, the elapsed time, was given (as in Example 2) by

$$s = t^2.$$  

Graphing $s$ as a function of $t \geq 0$ gives the right half of a parabola (Figure 10.1).

**SECTION 10.1 Limits and Motion: The Tangent Problem**

**EXPLORATION 1 Seeing Average Velocity**

Copy Figure 10.1 on a piece of paper and connect the points $(1, 1)$ and $(2, 4)$ with a straight line. (This is called a secant line because it connects two points on the curve.)

1. Find the slope of the line. 3
2. Find the average velocity of the ball over the time interval $[1, 2]$. 3 ft/sec
3. What is the relationship between the numbers that answer questions 1 and 2? They are the same.
4. In general, how could you represent the average velocity of the ball over the time interval $[a, b]$ geometrically?

 Exploration 1 suggests an important general fact: If $(a, s(a))$ and $(b, s(b))$ are two points on a distance-time graph, then the average velocity over the time interval $[a, b]$ can be thought of as the slope of the line connecting the two points. In fact, we designate both quantities with the same symbol: $\Delta s/\Delta t$.

Galileo knew this. He also knew that he wanted to find instantaneous velocity by letting the two points become one, resulting in $\Delta s/\Delta t = 0/0$, an algebraic impossibility. The picture, however, told a different story geometrically. If, for example, we were to connect pairs of points closer and closer to $(1, 1)$, our secant lines would look more and more like a line that is tangent to the curve at $(1, 1)$ (Figure 10.2).

It seemed obvious to Galileo and the other scientists of his time that the slope of the tangent line was the long-sought-after answer to the quest for instantaneous velocity. They could see it, but how could they compute it without dividing by zero? That was the “tangent line problem,” eventually solved for general functions by Newton and Leibniz in slightly different ways. We will solve it with limits as illustrated in Example 3.
EXAMPLE 3 Finding the Slope of a Tangent Line

Use limits to find the slope of the tangent line to the graph of \( s = t^2 \) at the point (1, 1) (Figure 10.2).

**SOLUTION** This will look a lot like the solution to Example 2.

\[
\lim_{t \to 1} \frac{\Delta s}{\Delta t} = \lim_{t \to 1} \frac{t^2 - 1^2}{t - 1} \\
= \lim_{t \to 1} \frac{(t + 1)(t - 1)}{t - 1} \\
= \lim_{t \to 1} (t + 1) \\
= 2
\]

Factor the numerator.

Since \( t \neq 1, \frac{t - 1}{t - 1} = 1 \)

Now try Exercise 17(a).

If you compare Example 3 to Example 2 it should be apparent that a method for solving the tangent line problem can be used to solve the instantaneous velocity problem, and vice versa. They are geometric and algebraic versions of the same problem!

**The Tangent Line Problem**

Although we have focused on Galileo’s work with motion problems in order to follow a coherent story, it was Pierre de Fermat (1601–1665) who first developed a “method of tangents” for general curves, recognizing its usefulness for finding relative maxima and minima. Fermat is best remembered for his work in number theory, particularly for Fermat’s Last Theorem, which states that there are no positive integers \( x, y, \) and \( z \) that satisfy the equation \( x^n + y^n = z^n \) if \( n \) is an integer greater than 2. Fermat wrote in the margin of a textbook, “I have a truly marvelous proof that this margin is too narrow to contain,” but if he had one, he apparently never wrote it down. Although mathematicians tried for over 330 years to prove (or disprove) Fermat’s Last Theorem, nobody succeeded until Andrew Wiles of Princeton University finally proved it in 1994.

**The Derivative**

Velocity, the rate of change of position with respect to time, is only one application of the general concept of “rate of change.” If \( y = f(x) \) is any function, we can speak of how \( y \) changes as \( x \) changes.

**Definition** Average Rate of Change

If \( y = f(x) \), then the average rate of change of \( y \) with respect to \( x \) on the interval \([a, b]\) is

\[
\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.
\]

Geometrically, this is the slope of the secant line through \((a, f(a))\) and \((b, f(b))\).

Using limits, we can proceed to a definition of the instantaneous rate of change of \( y \) with respect to \( x \) at the point where \( x = a \). This instantaneous rate of change is called the derivative.
A more computationally useful formula for the derivative is obtained by letting \( x = a + h \) and looking at the limit as \( h \) approaches 0 (equivalent to letting \( x \) approach \( a \)).

**DEFINITION Derivative at a Point (easier for computing)**

The derivative of the function \( f \) at \( x = a \), denoted by \( f'(a) \) and read “\( f \) prime of \( a \)” is

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},
\]

provided the limit exists.

Geometrically, this is the slope of the tangent line through \((a, f(a))\).

DIFFERENTIABILITY

We say a function is “differentiable” at \( a \) if \( f'(a) \) exists, because we can find the limit of the “quotient of differences.”

The fact that the derivative of a function at a point can be viewed geometrically as the slope of the line tangent to the curve \( y = f(x) \) at that point provides us with some insight as to how a derivative might fail to exist. Unless a function has a well-defined “slope” when you zoom in on it at \( a \), the derivative at \( a \) will not exist. For example, Figure 10.3 shows three cases for which \( f(0) \) exists but \( f'(0) \) does not.

**FIGURE 10.3** Three examples of functions defined at \( x = 0 \) but not differentiable at \( x = 0 \).
EXAMPLE 4 Finding a Derivative at a Point

Find $f'(4)$ if $f(x) = 2x^2 - 3$.

**SOLUTION**

$$f'(4) = \lim_{h \to 0} \frac{f(4 + h) - f(4)}{h}$$

$$= \lim_{h \to 0} \frac{2(4 + h)^2 - 3 - (2 \cdot 4^2 - 3)}{h}$$

$$= \lim_{h \to 0} \frac{2(16 + 8h + h^2) - 32}{h}$$

$$= \lim_{h \to 0} \frac{16h + 2h^2}{h}$$

$$= \lim_{h \to 0} (16 + 2h)$$

$$= 16$$

The derivative can also be thought of as a function of $x$. Its domain consists of all values in the domain of $f$ for which $f$ is differentiable. The function $f'$ can be defined by adapting the second definition above.

Now try Exercise 23.

**DEFINITION Derivative**

If $y = f(x)$, then the **derivative of the function $f$ with respect to $x$**, is the function $f'$ whose value at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

for all values of $x$ where the limit exists.

To emphasize the connection with slope $\triangle y/\triangle x$, Leibniz used the notation $dy/dx$ for the derivative. (The $dy$ and $dx$ were his “ghosts of departed quantities.”) This **Leibniz notation** has several advantages over the “prime” notation, as you will learn when you study calculus. We will use both notations in our examples and exercises.

EXAMPLE 5 Finding the Derivative of a Function

(a) Find $f'(x)$ if $f(x) = x^2$.

(b) Find $\frac{dy}{dx}$ if $y = \frac{1}{x}$.

continued
**SOLUTION**

(a) \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \]
\[ = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \]
\[ = \lim_{h \to 0} \frac{2xh + h^2}{h} \]
\[ = \lim_{h \to 0} (2x + h) \quad \text{Since } h \neq 0, \frac{h}{h} = 1. \]
\[ = 2x \]

So \( f'(x) = 2x \).

(b) \[ \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{1}{x + h} \cdot \frac{1}{x} \]
\[ = \lim_{h \to 0} \frac{x - (x + h)}{x(x + h)} \]
\[ = \lim_{h \to 0} \frac{1}{x} \cdot \frac{-h}{x(x + h)} \cdot \frac{1}{h} \]
\[ = \lim_{h \to 0} \frac{-1}{x(x + h)} \]
\[ = \frac{1}{x^2} \]

So \( \frac{dy}{dx} = -\frac{1}{x^2} \).

*Now try Exercise 29.*
CHAPTER OPENER PROBLEM
(from page 791)

PROBLEM: For an efficient windmill, the power generated in watts is given by the equation

\[ P = kr^2v^3 \]

where \( r \) is the radius of the propeller in meters, \( v \) is the wind velocity in meters per second, and \( k \) is a constant with units of \( \text{kg/m}^3 \). The exact value of \( k \) depends on various characteristics of the windmill.

Suppose a windmill has a propeller with radius 5 meters and \( k = 0.134 \text{ kg/m}^3 \).

(a) Find the function \( P(v) \) which gives power as a function of wind velocity.

(b) Find \( P'(7) \), the rate of change in power generated with respect to wind velocity, when the wind velocity is 7 meters per second.

SOLUTION:

(a) Since \( r = 5 \text{ m} \) and \( k = 0.134 \text{ kg/m}^3 \), we have

\[ P = kr^2v^3 = (0.134)(5^2)v^3 = 3.35v^3 \]

So, \( P(v) = 3.35v^3 \), where \( v \) is in meters per second and \( P \) is in watts.

(b) \( P'(7) = \lim_{h \to 0} \frac{P(7 + h) - P(7)}{h} \)

\[ = \lim_{h \to 0} \frac{3.35(7 + h)^3 - 3.35(7)^3}{h} \]

\[ = \lim_{h \to 0} \frac{3.35[(7^3 + 147h + 21h^2 + h^3) - 7^3]}{h} \]

\[ = \lim_{h \to 0} \frac{3.35(147h + 21h^2 + h^3)}{h} \]

\[ = \lim_{h \to 0} [3.35(147 + 21h + h^2)] \]

\[ = 3.35(147) \]

\[ = 492.45 \]

The rate of change in power generated is about 492 watts per meter/sec.
**SECTION 10.1 EXERCISES**

1. **Average Velocity** A bicyclist travels 21 miles in 1 hour and 45 minutes. What is her average velocity during the entire 1 3/4 hour time interval? 12 mi per hour

2. **Average Velocity** An automobile travels 540 kilometers in 4 hours and 30 minutes. What is its average velocity over the entire 4 1/2 hour time interval? 120 km per hour

In Exercises 3–6, the position of an object at time \( t \) is given by \( s(t) \). Find the instantaneous velocity at the indicated value of \( t \).

3. \( s(t) = 3t - 5 \) at \( t = 4 \) 3

4. \( s(t) = \frac{2}{t + 1} \) at \( t = 2 \) -2

5. \( s(t) = at^2 + 5 \) at \( t = 2 \) 4a

6. \( s(t) = \sqrt{t + 1} \) at \( t = 1 \) [Hint: “rationalize the numerator.”] \( 1/(2\sqrt{2}) \)

In Exercises 7–10, use the graph to estimate the slope of the tangent line, if it exists, to the graph at the given point.

7. \( x = 0 \) 1

8. \( x = 1 \) -1

9. \( x = 2 \) no tangent

10. \( x = 4 \) no tangent

In Exercises 11–14, graph the function in a square viewing window and, without doing any calculations, estimate the derivative of the function at the given point by interpreting it as the tangent line slope, if it exists at the point.

11. \( f(x) = x^2 - 2x + 5 \) at \( x = 3 \) 4

12. \( f(x) = \frac{1}{2}x^2 + 2x - 5 \) at \( x = 2 \) 4

13. \( f(x) = x^3 - 6x^2 + 12x - 9 \) at \( x = 0 \) 12

14. \( f(x) = 2 \sin x \) at \( x = \pi \) -2

15. **A Rock Toss** A rock is thrown straight up from level ground. The distance (in ft) the ball is above the ground (the position function) is \( f(t) = 3 + 48t - 16t^2 \) at any time \( t \) (in sec). Find

   (a) \( f'(0) \) 48

   (b) the initial velocity of the rock 48 ft/sec

16. **Rocket Launch** A toy rocket is launched straight up in the air from level ground. The distance (in ft) the rocket is above the ground (the position function) is \( f(t) = 170t - 16t^2 \) at any time \( t \) (in sec). Find

   (a) \( f'(0) \) 170

   (b) the initial velocity of the rocket 170 ft/sec
In Exercises 17–20, use the limit definition to find
(a) the slope of the graph of the function at the indicated point,
(b) an equation of the tangent line at the point.
(c) Sketch a graph of the curve near the point without using your
  graphing calculator.

17. \( f(x) = 2x^2 \) at \( x = -1 \)
18. \( f(x) = 2x - x^2 \) at \( x = 2 \)
19. \( f(x) = 2x^2 - 7x + 3 \) at \( x = 2 \)
20. \( f(x) = \frac{1}{x + 2} \) at \( x = 1 \)

In Exercises 21 and 22, estimate the slope of the tangent line to the
graph of the function, if it exists, at the indicated points.
21. \( f(x) = |x| \) at \( x = -2, 2, \) and 0. \(-1; 1; \) none
22. \( f(x) = \tan^{-1}(x + 1) \) at \( x = -2, 2, \) and 0. \( 0.5; 0.1; 0.5 \)

In Exercises 23–28, find the derivative, if it exists, of the function at
the specified point.
23. \( f(x) = 1 - x^2 \) at \( x = 2 \)
24. \( f(x) = 2x + \frac{1}{2}x^2 \) at \( x = 2 \)
25. \( f(x) = 3x^2 + 2 \) at \( x = -2 \)
26. \( f(x) = x^2 - 3x + 1 \) at \( x = 1 \)
27. \( f(x) = |x + 2| \) at \( x = -2 \)
28. \( f(x) = \frac{1}{x + 2} \) at \( x = -1 \)

In Exercises 29–32, find the derivative of \( f(x) \).
29. \( f(x) = 2 - 3x - 3 \)
30. \( f(x) = 2 - 3x^2 - 6x \)
31. \( f(x) = 3x^2 + 2x - 1 \)
32. \( f(x) = \frac{1}{x + 2} - \frac{1}{(x - 2)^2} \)

33. **Average Speed.** A lead ball is held at water level and dropped
   from a boat into a lake. The distance the ball falls at 0.1 sec time
   intervals is given in Table 10.1.

<table>
<thead>
<tr>
<th>Table 10.1 Distance Data of the Lead Ball</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (sec)</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
</tr>
<tr>
<td>0.3</td>
</tr>
<tr>
<td>0.4</td>
</tr>
<tr>
<td>0.5</td>
</tr>
<tr>
<td>0.6</td>
</tr>
<tr>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
</tr>
</tbody>
</table>

(a) Compute the average speed from 0.5 to 0.6 seconds and from
   0.8 to 0.9 seconds. \( 9 \) ft/sec; \( 15 \) ft/sec
(b) Find a quadratic regression model for the distance data and
   overlay its graph on a scatter plot of the data.
(c) Use the model in part (b) to estimate the depth of the lake if the
   ball hits the bottom after 2 seconds. \( 35.9 \) ft

34. **Finding Derivatives from Data.** A ball is dropped from the
   roof of a two-story building. The distance in feet above ground of
   the falling ball is given in Table 10.2 where \( t \) is in seconds.

<table>
<thead>
<tr>
<th>Table 10.2 Distance Data of the Ball</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (sec)</td>
</tr>
<tr>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
</tr>
<tr>
<td>0.6</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>1.0</td>
</tr>
<tr>
<td>1.2</td>
</tr>
<tr>
<td>1.4</td>
</tr>
</tbody>
</table>

(a) Use the data to estimate the average velocity of the ball in the
   interval \( 0.8 \leq t \leq 1. \) \( -27.4 \) ft/sec
(b) Find a quadratic regression model \( s \) for the data in Table 10.2
   and overlay its graph on a scatter plot of the data.
(c) Find the derivative of the regression equation and use it to estimate
   the velocity of the ball at time \( t = 1 \).

In Exercises 35–38, complete the following.
(a) Draw a graph of the function.
(b) Find the derivative of the function at the given point if it exists.
(c) Writing to Learn. If the derivative does not exist at the point,
   explain why not.
35. \( f(x) = \begin{cases} 4 - x & \text{if } x \leq 2 \\ x + 3 & \text{if } x > 2 \end{cases} \) at \( x = 2 \)
36. \( f(x) = \begin{cases} 1 + (x - 2)^2 & \text{if } x \leq 2 \\ 1 - (x - 2)^2 & \text{if } x > 2 \end{cases} \) at \( x = 2 \)
37. \( f(x) = \begin{cases} \frac{|x - 2|}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \) at \( x = 2 \)
38. \( f(x) = \begin{cases} \sin \frac{x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \) at \( x = 0 \)
In Exercises 39–42, sketch a possible graph for a function that has the stated properties.

39. The domain of \( f \) is \([0, 5]\) and the derivative at \( x = 2 \) is 3.
40. The domain of \( f \) is \([0, 5]\) and the derivative is 0 at both \( x = 2 \) and \( x = 4 \).
41. The domain of \( f \) is \([0, 5]\) and the derivative at \( x = 2 \) is undefined.
42. The domain of \( f \) is \([0, 5]\), \( f \) is nondecreasing on \([0, 5]\), and the derivative at \( x = 2 \) is 0.

Writing to Learn Explain why you can find the derivative of \( f(x) = ax + b \) without doing any computations. What is \( f'(x) \)? The slope of the line is \( a; f'(x) = a \)

Writing to Learn Use the definition of derivative at a point to express the derivative of \( f(x) = |x| \) at \( x = 0 \) as a limit. Then explain why the limit does not exist. (A graph of the quotient for \( x \) values near 0 might help.)

Standardized Test Questions

45. True or False When a ball rolls down a ramp, its instantaneous velocity is always zero. Justify your answer.
46. True or False If the derivative of the function \( f \) exists at \( x = a \), then the derivative is equal to the slope of the tangent line at \( x = a \). Justify your answer.

In Exercises 47–50, choose the correct answer. You may use a calculator.

47. Multiple Choice If \( f(x) = x^2 - 3x - 4 \), find \( f'(x) \).
   (A) \( x^2 + 3 \)  (B) \( x^2 - 4 \)  (C) \( 2x - 1 \)  (D) \( 2x + 3 \)
   (E) \( 2x - 3 \)
48. Multiple Choice If \( f(x) = 5x - 3x^2 \), find \( f'(x) \).
   (A) \( 5 - 6x \)  (B) \( 5 - 3x \)  (C) \( 5x - 6 \)  (D) \( 10x - 3 \)
   (E) \( 5x - 6x^2 \)
49. Multiple Choice If \( f(x) = x^3 \), find the derivative of \( f \) at \( x = 2 \).
   (A) 3  (B) 6  (C) 12  (D) 18  (E) Does not exist
50. Multiple Choice If \( f(x) = \frac{1}{x - 3} \), find the derivative of \( f \) at \( x = 1 \).
   (A) \( \frac{1}{4} \)  (B) \( \frac{1}{4} \)  (C) \( -\frac{1}{2} \)  (D) \( \frac{1}{2} \)  (E) Does not exist

Explorations

Graph each function in Exercises 51–54 and then answer the following questions.

(a) Writing to Learn Does the function have a derivative at \( x = 0 \)? Explain.

(b) Does the function appear to have a tangent line at \( x = 0 \)? If so, what is an equation of the tangent line?

Explorations

51. \( f(x) = |x| \)

52. \( f(x) = |x^{1/3}| \)

53. \( f(x) = x^{1/3} \)

54. \( f(x) = \tan^{-1} x \)

55. Free Fall A water balloon dropped from a window will fall a distance of \( s = 16t^2 \) feet during the first \( t \) seconds. Find the balloon’s (a) average velocity during the first 3 seconds of falling and (b) instantaneous velocity at \( t = 3 \).

56. Free Fall on Another Planet It can be established by experimentation that heavy objects dropped from rest free fall near the surface of another planet according to the formula \( y = gt^2 \), where \( y \) is the distance in meters the object falls in \( t \) seconds after being dropped. An object falls from the top of a 125 m spaceship which landed on the surface. It hits the surface in 5 seconds.
   (a) Find the value of \( g \). \( 5 \text{ m/sec}^2 \)
   (b) Find the average speed for the fall of the object. \( 25 \text{ m/sec} \)
   (c) With what speed did the object hit the surface? \( 50 \text{ m/sec} \)

Extending the Ideas

57. Graphing the Derivative The graph of \( f(x) = x^2 e^{-x} \) is shown below. Use your knowledge of the geometric interpretation of the derivative to sketch a rough graph of the derivative \( y = f'(x) \).

58. Group Activity The graph of \( y = f'(x) \) is shown below. Determine a possible graph for the function \( y = f(x) \).

59. Extending the Ideas

60. Writing to Learn Does the function have a derivative at \( x = 0 \)? Explain.

61. Does the function appear to have a tangent line at \( x = 0 \)? If so, what is an equation of the tangent line?
10.2
Limits and Motion: The Area Problem

What you'll learn about
- Distance from a Constant Velocity
- Distance from a Changing Velocity
- Limits at Infinity
- The Connection to Areas
- The Definite Integral

... and why
Like the tangent line problem, the area problem has many applications in every area of science, as well as historical and economic applications.

Distance from a Constant Velocity
“Distance equals rate times time” is one of the earliest problem-solving formulas that we learn in school mathematics. Given a velocity and a time period, we can use the formula to compute distance traveled—as in the following standard example.

EXAMPLE 1 Computing Distance Traveled
An automobile travels at a constant rate of 48 miles per hour for 2 hours and 30 minutes. How far does the automobile travel?

SOLUTION We apply the formula \( d = rt \):

\[
\begin{align*}
  d &= (48 \text{ mi/hr})(2.5 \text{ hr}) \\
  &= 120 \text{ miles}.
\end{align*}
\]

The similarity to Example 1 in Section 10.1 is intentional. In fact, if we represent distance traveled (i.e., the change in position) by \( s \) and the time interval by \( t \), the formula becomes

\[
\begin{align*}
  \Delta s &= (48 \text{ mph}) \Delta t,
\end{align*}
\]

which is equivalent to

\[
\frac{\Delta s}{\Delta t} = 48 \text{ mph}.
\]

So the two Example 1s are nearly identical—except that Example 1 of Section 10.1 did not make an assumption about constant velocity. What we computed in that instance was the average velocity over the 2.5-hour interval. This suggests that we could have actually solved the following, slightly different, problem to open this section.

EXAMPLE 2 Computing Distance Traveled
An automobile travels at an average rate of 48 miles per hour for 2 hours and 30 minutes. How far does the automobile travel?

SOLUTION The distance traveled is \( \Delta s \), the time interval has length \( \Delta t \), and \( \Delta s/\Delta t \) is the average velocity. Therefore,

\[
\begin{align*}
  \Delta s &= \frac{\Delta s}{\Delta t} \cdot \Delta t \\
  &= (48 \text{ mph})(2.5 \text{ hr}) \\
  &= 120 \text{ miles}.
\end{align*}
\]

So, given average velocity over a time interval, we can easily find distance traveled. But suppose we have a velocity function \( v(t) \) that gives instantaneous velocity as a changing function of time. How can we use the instantaneous velocity function to find distance...
traveled over a time interval? This was the other intriguing problem about instantaneous velocity that puzzled the 17th-century scientists—and once again, algebra was inadequate for solving it, as we shall see.

**Distance from a Changing Velocity**

When Galileo began his experiments, here’s what he might have asked himself about using a changing velocity to find distance:

One might be tempted to offer the following “solution”:

Velocity times \( \Delta t \) gives \( \Delta s \). But instantaneous velocity occurs at an instant of time, so \( \Delta r = 0 \). That means \( \Delta s = 0 \). So, at any given instant of time, the ball doesn’t move. Since any time interval consists of instants of time, the ball never moves at all! (You might well ask: Is this another trick question?)

As was the case with the Velocity Question in Section 10.1, this foolish-looking example conceals a very subtle algebraic dilemma—and, far from being a trick question, it is exactly the question that needed to be answered in order to compute the distance traveled by an object whose velocity varies as a function of time. The scientists who were working on the tangent line problem realized that the distance-traveled problem must be related to it, but, surprisingly, their geometry led them in another direction. The distance traveled problem led them not to tangent lines, but to areas.

**Limits at Infinity**

Before we see the connection to areas, let us revisit another limit concept that will make instantaneous velocity easier to handle, just as in the last section. We will again be content with an informal definition.

**EXPLORATION 1 ** An Infinite Limit

A gallon of water is divided equally and poured into teacups. Find the amount in each teacup and the total amount in all the teacups if there are

1. 10 teacups  0.1 gal; 1 gal
2. 100 teacups  0.01 gal; 1 gal
3. 1 billion teacups  0.000000001 gal; 1 gal
4. an infinite number of teacups  0 gal; 1 gal
The preceding Exploration probably went pretty smoothly until you came to the infinite number of teacups. At that point you were probably pretty comfortable in saying what the total amount would be, and probably a little uncomfortable in saying how much would be in each teacup. (Theoretically it would be zero, which is just one reason why the actual experiment cannot be performed.) In the language of limits, the total amount of water in the infinite number of teacups would look like this:

$$\lim_{n \to \infty} \frac{n \cdot \frac{1}{n}}{\frac{1}{n}} = \lim_{n \to \infty} n = 1 \text{ gallon}$$

while the total amount in each teacup would look like this:

$$\lim_{n \to \infty} \frac{1}{n} = 0 \text{ gallons.}$$

Summing up an infinite number of nothings to get something is mysterious enough when we use limits; without limits it seems to be an algebraic impossibility. That is the dilemma that faced the 17th-century scientists who were trying to work with instantaneous velocity. Once again, it was geometry that showed the way when the algebra failed.

### The Connection to Areas

If we graph the constant velocity \( v = 48 \) in Example 1 as a function of time \( t \), we notice that the area of the shaded rectangle is the same as the distance traveled (Figure 10.4). This is no mere coincidence, either, as the area of the rectangle and the distance traveled over the time interval are both computed by multiplying the same two quantities:

$$(48 \text{ mph})(2.5 \text{ hr}) = 120 \text{ miles.}$$

Now suppose we graph a velocity function that varies continuously as a function of time (Figure 10.5). Would the area of this irregularly-shaped region still give the total distance traveled over the time interval \([a, b]\)?

Newton and Leibniz (and, actually, many others who had considered this question) were convinced that it obviously would, and that is why they were interested in a calculus for finding areas under curves. They imagined the time interval being partitioned into many tiny subintervals, each one so small that the velocity over it would essentially be constant. Geometrically, this was equivalent to slicing the area into narrow strips, each one of which would be nearly indistinguishable from a narrow rectangle (Figure 10.6).

The idea of partitioning irregularly-shaped areas into approximating rectangles was not new. Indeed, Archimedes had used that very method to approximate the area of a circle with remarkable accuracy. However, it was an exercise in patience and perseverance, as Example 3 will show.
EXAMPLE 3  Approximating an Area with Rectangles

Use the six rectangles in Figure 10.7 to approximate the area of the region below the graph of \( f(x) = x^2 \) over the interval \([0, 3]\).

**SOLUTION** The base of each approximating rectangle is \( \frac{1}{2} \). The height is determined by the function value at the right-hand endpoint of each subinterval. The areas of the six rectangles and the total area are computed in the table below:

<table>
<thead>
<tr>
<th>Subinterval</th>
<th>Base of Rectangle</th>
<th>Height of Rectangle</th>
<th>Area of Rectangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 1/2])</td>
<td>(1/2)</td>
<td>(f(1/2) = (1/2)^2 = 1/4)</td>
<td>((1/2)(1/4) = 0.125)</td>
</tr>
<tr>
<td>([1/2, 1])</td>
<td>(1/2)</td>
<td>(f(1) = (1)^2 = 1)</td>
<td>((1/2)(1) = 0.500)</td>
</tr>
<tr>
<td>([1, 3/2])</td>
<td>(1/2)</td>
<td>(f(3/2) = (3/2)^2 = 9/4)</td>
<td>((1/2)(9/4) = 1.125)</td>
</tr>
<tr>
<td>([3/2, 2])</td>
<td>(1/2)</td>
<td>(f(2) = (2)^2 = 4)</td>
<td>((1/2)(4) = 2.000)</td>
</tr>
<tr>
<td>([2, 5/2])</td>
<td>(1/2)</td>
<td>(f(5/2) = (5/2)^2 = 25/4)</td>
<td>((1/2)(25/4) = 3.125)</td>
</tr>
<tr>
<td>([5/2, 3])</td>
<td>(1/2)</td>
<td>(f(3) = (3)^2 = 9)</td>
<td>((1/2)(9) = 4.500)</td>
</tr>
</tbody>
</table>

**Total Area:** \(11.375\)

The six rectangles give a (rather crude) approximation of 11.375 square units for the area under the curve from 0 to 3.

Figure 10.7 shows that the right rectangular approximation method (RRAM) in Example 4 overestimates the true area. If we were to use the function values at the left-hand endpoints of the subintervals (LRAM), we would obtain a rectangular approximation (6.875 square units) that underestimates the true area (Figure 10.8). The average of the two approximations is 9.125 square units, which is actually a pretty good estimate of the true area of 9 square units. If we were to repeat the process with 20 rectangles, the average would be 9.01125. This method of converging toward an unknown area by refining approximations is tedious, but it works—Archimedes used a variation of it 2200 years ago to estimate the area of a circle, and in the process demonstrated that the ratio of the circumference to the diameter was between 3.140845 and 3.142857.

The calculus step is to move from a finite number of rectangles (yielding an approximate area) to an infinite number of rectangles (yielding an exact area). This brings us to the definite integral.
RIEMANN SUMS

A sum of the form \( \sum_{i=1}^{n} f(x_i) \Delta x \) in which \( x_1 \) is in the first subinterval, \( x_2 \) is in the second, and so on, is called a Riemann sum, in honor of Georg Riemann (1826–1866), who determined the functions for which such sums had limits as \( n \to \infty \).

DEFINITE INTEGRAL NOTATION

Notice that the notation for the definite integral (another legacy of Leibniz) parallels the sigma notation of the sum for which it is a limit. The “\( \Sigma \)” in the limit becomes a stylized “\( S \),” for “sum.” The “\( \Delta x \)” becomes “\( dx \)” (as it did in the derivative), and the “\( f(x) \)” becomes simply “\( f(x) \)” because we are effectively summing up all the \( f(x) \) values along the interval (times an arbitrarily small change in \( x \)), rendering the subscripts unnecessary.

The Definite Integral

In general, begin with a continuous function \( y = f(x) \) over an interval \([a, b]\). Divide \([a, b]\) into \( n \) subintervals of length \( \Delta x = (b - a)/n \). Choose any value \( x_1 \) in the first subinterval, \( x_2 \) in the second, and so on. Compute \( f(x_1), f(x_2), f(x_3), \ldots, f(x_n) \), multiply each value by \( \Delta x \), and sum up the products. In sigma notation, the sum of the products is

\[
\sum_{i=1}^{n} f(x_i) \Delta x.
\]

The limit of this sum as \( n \) approaches infinity is the solution to the area problem, and hence the solution to the problem of distance traveled. Indeed, it solves a variety of other problems as well, as you will learn when you study calculus. The limit, if it exists, is called a definite integral.

**DEFINITION** Definite Integral

Let \( f \) be a function defined on \([a, b]\) and let \( \sum_{i=1}^{n} f(x_i) \Delta x \) be defined as above. The definite integral of \( f \) over \([a, b]\), denoted \( \int_{a}^{b} f(x) \, dx \), is given by

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x,
\]

provided the limit exists. If the limit exists, we say \( f \) is integrable on \([a, b]\).

The solution to Example 4 shows that it can be tedious to approximate a definite integral by working out the sum for a large value of \( n \). One of the crowning achievements of calculus was to demonstrate how the exact value of a definite integral could be obtained without summing up any products at all. You will have to wait until calculus to see how that is done; meanwhile, you will learn in Section 10.4 how to use a calculator to take the tedium out of finding definite integrals by summing.

You can also use the area connection to your advantage, as shown in these next two examples.

**EXAMPLE 4** Computing an Integral

Find \( \int_{1}^{5} 2x \, dx \).

**SOLUTION** This will be the area under the line \( y = 2x \) over the interval \([1, 5]\). The graph in Figure 10.9 shows that this is the area of a trapezoid.

Using the formula \( A = \frac{h}{2} (b_1 + b_2) \), we find that

\[
\int_{1}^{5} 2x \, dx = 4 \left( \frac{2(1) + 2(5)}{2} \right) = 24
\]

Now try Exercise 23.
EXAMPLE 5 Computing an Integral

Suppose a ball rolls down a ramp so that its velocity after \( t \) seconds is always \( 2t \) feet per second. How far does it fall during the first 3 seconds?

**SOLUTION**

The distance traveled will be the same as the area under the velocity graph, \( v(t) = 2t \), over the interval \([0, 3]\). The graph is shown in Figure 10.10. Since the region is triangular, we can find its area:

\[
A = \frac{1}{2} \times 3 \times 6 = 9. 
\]

The distance traveled in the first 3 seconds, therefore, is:

\[ s = \frac{1}{2} \times 3 \times 6 = 9 \text{ feet}. \]

Now try Exercise 45.

**SECTION 10.2 Limits and Motion: The Area Problem**

**QUICK REVIEW 10.2** (For help, go to Sections 1.1 and 9.4.)

In Exercises 1 and 2, list the elements of the sequence.

1. \( a_k = \frac{1}{2} \left( \frac{1}{2} k \right)^2 \) for \( k = 1, 2, 3, 4, \ldots, 9, 10 \)

2. \( a_k = \frac{1}{4} \left( 2 + \frac{1}{4} k \right)^2 \) for \( k = 1, 2, 3, 4, \ldots, 9, 10 \)

In Exercises 3–6, find the sum.

3. \( \sum_{k=1}^{10} \frac{1}{2} (k + 1) = \frac{65}{2} \)

4. \( \sum_{k=1}^{n} (k + 1) = \frac{n(n + 3)}{2} \)

5. \( \sum_{k=1}^{10} \frac{1}{2} (k + 1)^2 = \frac{505}{2} \)

6. \( \sum_{k=1}^{n} \frac{1}{2} k^2 = \frac{n(n + 1)(2n + 1)}{12} \)

7. A truck travels at an average speed of 57 mph for 4 hours. How far does it travel? 228 miles

8. A pump working at 5 gal/min pumps for 2 hours. How many gallons are pumped? 600 gal

9. Water flows over a spillway at a steady rate of 200 cubic feet per second. How many cubic feet of water pass over the spillway in 6 hours? 4,320,000 ft³

10. A county has a population density of 560 people per square mile in an area of 35,000 square miles. What is the population of the county? 19,600,000 people
SECTION 10.2 EXERCISES

In Exercises 1–4, explain how to represent the problem situation as an area question and then solve the problem.

1. A train travels at 65 mph for 3 hours. How far does it travel? 195 mi

2. A pump working at 15 gal/min pumps for one-half hour. How many gallons are pumped? 450 gal

3. Water flows over a spillway at a steady rate of 150 cubic feet per second. How many cubic feet of water pass over the spillway in one hour? 540,000 ft³

4. A city has a population density of 650 people per square mile in an area of 20 square miles. What is the population of the city? 13,000 people

5. An airplane travels at an average velocity of 640 kilometers per hour for 3 hours and 24 minutes. How far does the airplane travel? 2176 km

6. A train travels at an average velocity of 24 miles per hour for 4 hours and 50 minutes. How far does the train travel? 116 mi

In Exercises 7–10, estimate the area of the region above the x-axis and under the graph of the function from x = 0 to x = 5.

7. ![Graph 1]

8. ![Graph 2]

9. ![Graph 3]

10. ![Graph 4]

In Exercises 11 and 12, use the 8 rectangles shown to approximate the area of the region below the graph of f(x) = 10 − x² over the interval [−1, 3].

11. 32.5

12. 28.5

In Exercises 13–16, partition the given interval into the indicated number of subintervals.

13. [0, 2]; 4

14. [0, 2]; 8

15. [1, 4]; 6

16. [1, 5]; 8

In Exercises 17–20, complete the following.

(a) Draw the graph of the function for x in the specified interval. Verify that the function is nonnegative in that interval.

(b) On the graph in part (a), draw and shade the approximating rectangles for the RRAM using the specified partition. Compute the RRAM area estimate without using a calculator.

(c) Repeat part (b) using the LRAM.

(d) Average the RRAM and LRAM approximations from parts (b) and (c) to find an average estimate of the area.

17. f(x) = x²; [0, 4]; 4 subintervals

18. f(x) = x³ + 2; [0, 6]; 6 subintervals

19. f(x) = 4x − x²; [0, 4]; 4 subintervals

20. f(x) = x²; [0, 3]; 3 subintervals

In Exercises 21–28, find the definite integral by computing an area. (It may help to look at a graph of the function.)

21. \( \int_{-1}^{3} 5 \, dx \)

22. \( \int_{-1}^{6} 6 \, dx \)

23. \( \int_{0}^{4} 3x \, dx \)

24. \( \int_{0}^{1} 0.5x \, dx \)

25. \( \int_{1}^{5} (x + 3) \, dx \)

26. \( \int_{1}^{4} (3x - 2) \, dx \)

27. \( \int_{0}^{2} \sqrt{4 - x^2} \, dx \)

28. \( \int_{0}^{3} \sqrt{36 - x^2} \, dx \)
It can be shown that the area enclosed between the x-axis and one arch of the sine curve is 2. Use this fact in Exercises 29–38 to compute the definite integral. (It may help to look at a graph of the function.)

29. \[ \int_{0}^{\pi} \sin x \, dx \]
30. \[ \int_{0}^{\pi} (\sin x + 2) \, dx \]
31. \[ \int_{0}^{\pi} \sin (x + 2) \, dx \]
32. \[ \int_{0}^{\pi} \cos x \, dx \]
33. \[ \int_{0}^{\pi} \sin x \, dx \]
34. \[ \int_{0}^{\frac{\pi}{2}} \cos x \, dx \]
35. \[ \int_{0}^{\pi} 2 \sin x \, dx \] [Hint: All the rectangles are twice as tall.]
36. \[ \int_{0}^{\pi} 2 \sin \left(\frac{x}{2}\right) \, dx \] [Hint: All the rectangles are twice as wide.]
37. \[ \int_{0}^{\pi} |\sin x| \, dx \]
38. \[ \int_{0}^{\pi} |\cos x| \, dx \]

In Exercises 39–42, find the integral assuming that \( k \) is a number between 0 and 4.

39. \[ \int_{0}^{k} (kx + 3) \, dx \]
40. \[ \int_{0}^{k} (4x + 3) \, dx \]
41. \[ \int_{0}^{4} (3x + k) \, dx \]
42. \[ \int_{0}^{4} (4x + 3) \, dx \]

43. **Writing to Learn** Let \( g(x) = -f(x) \) where \( f \) has nonnegative function values on an interval \([a, b] \). Explain why the area above the graph of \( g \) is the same as the area under the graph of \( f \) in the same interval.

44. **Writing to Learn** Explain how you can find the area under the graph of \( f(x) = \sqrt{16 - x^2} \) from \( x = 0 \) to \( x = 4 \) by mental computation only.

45. **Falling Ball** Suppose a ball is dropped from a tower and its velocity after \( t \) seconds is always 32t feet per second. How far does the ball fall during the first 2 seconds? 64 ft

46. **Accelerating Automobile** Suppose an automobile accelerates so that its velocity after \( t \) seconds is always 6t feet per second. How far does the car travel in the first 7 seconds? 147 ft

47. **Rock Toss** A rock is thrown straight up from level ground. The velocity of the rock at any time \( t \) (sec) is \( v(t) = 48 - 32t \) ft/sec.
(a) Graph the velocity function.
(b) At what time does the rock reach its maximum height?
(c) Find how far the rock has traveled at its maximum height.

48. **Rocket Launch** A toy rocket is launched straight up from level ground. Its velocity function is \( f(t) = 170 - 32t \) feet per second, where \( t \) is the number of seconds after launch.
(a) Graph the velocity function.
(b) At what time does the rocket reach its maximum height?
(c) Find how far the rocket has traveled at its maximum height. \( \approx 451.6 \) ft

### Table 10.3 Velocity Data of the Ball

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Velocity (ft/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-5.05</td>
</tr>
<tr>
<td>0.4</td>
<td>-11.43</td>
</tr>
<tr>
<td>0.6</td>
<td>-17.46</td>
</tr>
<tr>
<td>0.8</td>
<td>-24.21</td>
</tr>
<tr>
<td>1.0</td>
<td>-30.62</td>
</tr>
<tr>
<td>1.2</td>
<td>-37.06</td>
</tr>
<tr>
<td>1.4</td>
<td>-43.47</td>
</tr>
</tbody>
</table>

(a) Draw a scatter plot of the data.
(b) Find the approximate building height using RRAM areas as in Example 4. Use the fact that if the velocity function is always negative the distance traveled will be the same as if the absolute value of the velocity values were used. 33.86 ft

### Table 10.4 Weight of a Leaking Water Barrel

<table>
<thead>
<tr>
<th>Distance (ft)</th>
<th>Weight (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1250</td>
</tr>
<tr>
<td>5</td>
<td>1150</td>
</tr>
<tr>
<td>10</td>
<td>1050</td>
</tr>
<tr>
<td>15</td>
<td>950</td>
</tr>
<tr>
<td>20</td>
<td>850</td>
</tr>
<tr>
<td>25</td>
<td>750</td>
</tr>
<tr>
<td>30</td>
<td>650</td>
</tr>
</tbody>
</table>

50. **Work** Work is defined as force times distance. A full water barrel weighing 1250 pounds has a significant leak and must be lifted 35 feet. Table 10.4 displays the weight of the barrel measured after each 5 feet of movement. Find the approximate work in foot-pounds done in lifting the barrel 35 feet. 31,500 ft-pounds

### Standardized Test Questions

51. **True or False** When estimating the area under a curve using LRAM, the accuracy typically improves as the number \( n \) of subintervals is increased.

52. **True or False** The statement \( \lim_{x \to L} f(x) = L \) means that \( f(x) \) gets arbitrarily large as \( x \) gets arbitrarily close to \( L \).
It can be shown that the area of the region enclosed by the curve \( y = \sqrt{x} \), the \( x \)-axis, and the line \( x = 9 \) is 18. Use this fact in Exercises 53–56 to choose the correct answer. Do not use a calculator.

53. Multiple Choice \( \int_{0}^{9} 2\sqrt{x} \, dx \)
   (A) 36  (B) 27  (C) 18  (D) 9  (E) 6

54. Multiple Choice \( \int_{0}^{9} (\sqrt{x} + 5) \, dx \)
   (A) 14  (B) 23  (C) 33  (D) 45  (E) 63

55. Multiple Choice \( \int_{1}^{14} (\sqrt{x} - 5) \, dx \)
   (A) 9  (B) 13  (C) 18  (D) 23  (E) 28

56. Multiple Choice \( \int_{0}^{3} \sqrt{3x} \, dx \)
   (A) 54  (B) 18  (C) 9  (D) 6  (E) 3

58. Area Under a Discontinuous Function Let
   \[ f(x) = \begin{cases} 
   1 & \text{if } x < 2 \\
   x & \text{if } x > 2 
   \end{cases} \]

   (a) Draw a graph of \( f \). Determine its domain and range.
   (b) Writing to Learn How would you define the area under \( f \) from \( x = 0 \) to \( x = 4 \)? Does it make a difference if the function has no value at \( x = 2 \)?

Extending the Ideas

Group Activity From what you know about definite integrals, decide whether each of the following statements is true or false for integrable functions (in general). Work with your classmates to justify your answers.

59. \( \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx = \int_{a}^{b} (f(x) + g(x)) \, dx \) true

60. \( \int_{a}^{b} 8 \cdot f(x) \, dx = 8 \cdot \int_{a}^{b} f(x) \, dx \) true

61. \( \int_{a}^{b} f(x) \cdot g(x) \, dx = \int_{a}^{b} f(x) \, dx \cdot \int_{a}^{b} g(x) \, dx \) false

62. \( \int_{a}^{b} f(x) \, dx + \int_{c}^{d} f(x) \, dx = \int_{a}^{d} f(x) \, dx \) for \( a < c < b \) true

63. \( \int_{a}^{b} f(x) \, dx = \int_{b}^{a} f(x) \) false

64. \( \int_{a}^{b} f(x) \, dx = 0 \) true

Explorations

57. Group Activity You may have erroneously assumed that the function \( f \) had to be positive in the definition of the definite integral. It is a fact that \( \int_{0}^{2\pi} \sin x \, dx = 0 \). Use the definition of the definite integral to explain why this is so. What does this imply about \( \int_{0}^{1} (x - 1) \, dx \)?

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A Little History

Progress in mathematics occurs gradually and without much fanfare in the early stages. The fanfare occurs much later, after the discoveries and innovations have been cleaned up and put into perspective. Calculus is certainly a case in point. Most of the ideas in this chapter pre-date Newton and Leibniz. Others were solving calculus problems as far back as Archimedes of Syracuse (ca. 287–212 B.C.), long before calculus was “discovered.” What Newton and Leibniz did was to develop the rules of the game, so that derivatives and integrals could be computed algebraically. Most importantly, they developed what has come to be called the Fundamental Theorem of Calculus, which explains the connection between the “tangent line problem” and the “area problem.”

But the methods of Newton and Leibniz depended on mysterious “infinitesimal” quantities that were small enough to vanish and yet were not zero. Jean Le Rond d’Alembert (1717–1783) was a strong proponent of replacing infinitesimals with limits (the strategy that would eventually work), but these concepts were not well understood until Karl Weierstrass (1815–1897) and his student Eduard Heine (1821–1881) introduced the formal, unassailable definitions that are used in our higher mathematics courses today. By that time, Newton and Leibniz had been dead for over 150 years.

Defining a Limit Informally

There is nothing difficult about the following limit statements:

\[
\lim_{x \to 2} (2x - 1) = 5 \quad \lim_{x \to \infty} (x^2 + 3) = \infty \quad \lim_{n \to \infty} \frac{1}{n} = 0
\]

That is why we have used limit notation throughout this book. Particularly when electronic graphers are available, analyzing the limiting behavior of functions algebraically, numerically, and graphically can tell us much of what we need to know about the functions.

What is difficult is to come up with an air-tight definition of what a limit really is. If it had been easy, it would not have taken 150 years. The subtleties of the “epsilon-delta” definition of Weierstrass and Heine are as beautiful as they are profound, but they are not the stuff of a precalculus course. Therefore, even as we look more closely at limits and their properties in this section, we will continue to refer to our “informal” definition of limit (essentially that of d’Alembert). We repeat it here for ready reference:

**Definition (Informal) Limit at** \(a\)

When we write “\(\lim_{x \to a} f(x) = L\)” we mean that \(f(x)\) gets arbitrarily close to \(L\) as \(x\) approaches \(a\) arbitrarily close (but not equal) to \(a\).
EXPLORATION 1  What’s the Limit?

As a class, discuss the following two limit statements until you really understand why they are true. Look at them every way you can. Use your calculators. Do you see how the above definition verifies that they are true? In particular, can you defend your position against the challenges that follow the statements? (This exploration is intended to be free-wheeling and philosophical. You can’t prove these statements without a stronger definition.)

1. \( \lim_{x \to 2} \frac{7}{x} = 14.000000000000000001 \)

Challenges:
- Isn’t \( 7x \) getting “arbitrarily close” to that number as \( x \) approaches 2?
- How can you tell that 14 is the limit and 14.000000000000000001 is not?

2. \( \lim_{x \to 0} \frac{x^2 + 2x}{x} = 2 \)

Challenges:
- How can the limit be 2 when the quotient isn’t even defined at 0?
- Won’t there be an asymptote at \( x = 0 \)? The denominator equals 0 there.
- How can you tell that 2 is the limit and 1.99999999999999999999 is not?

EXPLORATION EXTENSIONS

Discuss the following limit statement and decide whether it is true:

\[ \lim_{x \to 0} x \sin \frac{1}{x} = 0. \]

EXAMPLE 1  Finding a Limit

Find \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \).

**Solve Graphically**
The graph in Figure 10.11a suggests that the limit exists and is about 3.

[Graph showing \( f(x) = \frac{x^3 - 1}{x - 1} \) with values: \( X=1.0212766 \Rightarrow Y=3.0642825 \).]

**Solve Numerically**
The table also gives compelling evidence that the limit is 3.

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</tr>
<tr>
<td>.99999</td>
<td>3.003</td>
</tr>
<tr>
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<td>3.009</td>
</tr>
<tr>
<td>1.003</td>
<td>3.009</td>
</tr>
</tbody>
</table>

**Solve Algebraically**

\[ \lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} \]
\[ = \lim_{x \to 1} (x^2 + x + 1) \]
\[ = 1 + 1 + 1 \]
\[ = 3 \]

**FIGURE 10.11a**  A graph of \( f(x) = \frac{x^3 - 1}{x - 1} \).

As convincing as the graphical and numerical evidence is, the best evidence is algebraic. The limit is 3.

Now try Exercise 11.
Properties of Limits

When limits exist, there is nothing unusual about the way they interact algebraically with each other. You could easily predict that the following properties would hold. These are all theorems that one could prove with a rigorous definition of limit, but we must state them without proof here.

### Properties of Limits

If \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) both exist, then

1. **Sum Rule**
   \[
   \lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)
   \]

2. **Difference Rule**
   \[
   \lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)
   \]

3. **Product Rule**
   \[
   \lim_{x \to c} (f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)
   \]

4. **Constant Multiple Rule**
   \[
   \lim_{x \to c} (k \cdot g(x)) = k \cdot \lim_{x \to c} g(x)
   \]

5. **Quotient Rule**
   \[
   \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)},
   \quad \text{provided } \lim_{x \to c} g(x) \neq 0
   \]

6. **Power Rule**
   \[
   \lim_{x \to c} (f(x))^n = (\lim_{x \to c} f(x))^n \quad \text{for } n \text{ a positive integer}
   \]

7. **Root Rule**
   \[
   \lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} \quad \text{for } n \geq 2
   \]
   \quad \text{a positive integer, provided } \sqrt[n]{\lim_{x \to c} f(x)}
   \quad \text{and } \lim_{x \to c} \sqrt[n]{f(x)} \text{ are real numbers.}

### EXAMPLE 2  Using the Limit Properties

You will learn in Example 10 that
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]
Use this fact, along with the limit properties, to find the following limits:

(a) \[
\lim_{x \to 0} \frac{x + \sin x}{x}
\]
(b) \[
\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2}
\]
(c) \[
\lim_{x \to 0} \frac{\sqrt[3]{\sin x}}{\sqrt{x}}
\]

### SOLUTION

(a) \[
\lim_{x \to 0} \frac{x + \sin x}{x} = \lim_{x \to 0} \left( \frac{x}{x} + \frac{\sin x}{x} \right)
\]
   \[
   = \lim_{x \to 0} \frac{x}{x} + \lim_{x \to 0} \frac{\sin x}{x} \quad \text{Sum Rule}
   \]
   \[
   = 1 + 1
   \]
   \[
   = 2
   \]

continued
\[
\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2} = \lim_{x \to 0} \frac{\sin^2 x}{x^2} \\
= \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^2 \\
= \left( \lim_{x \to 0} \frac{\sin x}{x} \right)^2 \quad \text{Pythagorean identity} \\
= 1^2 \\
= 1 \\
\]

\[
\lim_{x \to 0} \frac{\sqrt{\sin x}}{\sqrt{x}} = \lim_{x \to 0} \sqrt{\frac{\sin x}{x}} \\
= \sqrt{\lim_{x \to 0} \frac{\sin x}{x}} \quad \text{Root Rule} \\
= \sqrt{1} \\
= 1 \\
\]

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**Limits of Continuous Functions**

Recall from Section 1.2 that a function is continuous at \( a \) if \( \lim_{x \to a} f(x) = f(a) \). This means that the limit (at \( a \)) of a function can be found by “plugging in \( a \)” provided the function is continuous at \( a \). (The condition of continuity is essential when employing this strategy. For example, plugging in 0 does not work on any of the limits in Example 2.)

**EXAMPLE 3** Finding Limits by Substitution

Find the limits.

(a) \( \lim_{x \to 0} \frac{e^x - \tan x}{\cos^2 x} \)

(b) \( \lim_{n \to 16} \frac{\sqrt{n}}{\log_2 n} \)

**SOLUTION**

You might not recognize these functions as being continuous, but you can use the limit properties to write the limits in terms of limits of basic functions.

(a) \( \lim_{x \to 0} \frac{e^x - \tan x}{\cos^2 x} = \frac{\lim_{x \to 0} (e^x - \tan x)}{\lim_{x \to 0} (\cos^2 x)} \quad \text{Quotient Rule} \\
\quad = \frac{\lim_{x \to 0} e^x - \lim_{x \to 0} \tan x}{\lim_{x \to 0} (\cos x)^2} \quad \text{Difference and Power Rules} \\
\quad = \frac{e^0 - \tan 0}{(\cos 0)^2} \quad \text{Limits of continuous functions} \\
\quad = \frac{1 - 0}{1} \\
\quad = 1 \\
\)
Example 3 hints at some important properties of continuous functions that follow from the properties of limits. If \( f \) and \( g \) are both continuous at \( x = a \), then so are \( f + g \), \( f - g \), \( fg \), and \( f/g \) (with the assumption that \( g(a) \) does not create a zero denominator in the quotient). Also, the \( n \)th power and \( n \)th root of a function that is continuous at \( a \) will also be continuous at \( a \) (with the assumption that \( \sqrt[n]{f(a)} \) is real).

**One-sided and Two-sided Limits**

We can see that the limit of the function in Figure 10.11 is 3 whether \( x \) approaches 1 from the left or right. Sometimes the values of a function \( f \) can approach different values as \( x \) approaches a number \( c \) from opposite sides. When this happens, the limit of \( f \) as \( x \) approaches \( c \) from the left is the left-hand limit of \( f \) at \( c \) and the limit of \( f \) as \( x \) approaches \( c \) from the right is the right-hand limit of \( f \) at \( c \). Here is the notation we use:

| Left-hand: \( \lim_{x \to c^-} f(x) \) | The limit of \( f \) as \( x \) approaches \( c \) from the left. |
| Right-hand: \( \lim_{x \to c^+} f(x) \) | The limit of \( f \) as \( x \) approaches \( c \) from the right. |

**EXAMPLE 4 Finding Left- and Right-Hand Limits**

Find \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \) where \( f(x) = \begin{cases} -x^2 + 4x - 1 & \text{if } x \leq 2 \\ 2x - 3 & \text{if } x > 2 \end{cases} \)

**SOLUTION** Figure 10.12 suggests that the left- and right-hand limits of \( f \) exist but are not equal. Using algebra we find:

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-x^2 + 4x - 1) = -2^2 + 4 \cdot 2 - 1 = 3
\]

\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 2 \cdot 2 - 3 = 1
\]

You can use trace or tables to support the above results.

Now try Exercise 27, parts (a) and (b).
The limit \( \lim_{x \to c} f(x) \) is sometimes called the two-sided limit of \( f \) at \( c \) to distinguish it from the one-sided left-hand and right-hand limits of \( f \) at \( c \). The following theorem indicates how these limits are related.

**THEOREM One-sided and Two-sided Limits**

A function \( f(x) \) has a limit as \( x \) approaches \( c \) if and only if the left-hand and right-hand limits at \( c \) exist and are equal. That is,

\[
\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.
\]

The limit of the function \( f \) of Example 4 as \( x \) approaches 2 does not exist, so \( f \) is discontinuous at \( x = 2 \). However, discontinuous functions can have a limit at a point of discontinuity. The function \( f \) of Example 1 is discontinuous at \( x = 1 \) because \( f(1) \) does not exist, but it has the limit 3 as \( x \) approaches 1. Example 5 illustrates another way a function can have a limit and still be discontinuous.

**EXAMPLE 5 Finding a Limit at a Point of Discontinuity**

Let

\[
f(x) = \begin{cases} 
\frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\
2 & \text{if } x = 3.
\end{cases}
\]

Find \( \lim_{x \to 3} f(x) \) and prove that \( f \) is discontinuous at \( x = 3 \).

**SOLUTION** Figure 10.13 suggests that the limit of \( f \) as \( x \) approaches 3 exists. Using algebra we find

\[
\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6.
\]

Because \( f(3) = 2 \neq \lim_{x \to 3} f(x) \), \( f \) is discontinuous at \( x = 3 \).

**EXAMPLE 6 Finding One-Sided and Two-Sided Limits**

Let \( f(x) = \text{int}(x) \) (the greatest integer function). Find:

\[
(a) \lim_{x \to 3^-} \text{int}(x) \quad (b) \lim_{x \to 3^+} \text{int}(x) \quad (c) \lim_{x \to 3} \text{int}(x)
\]

continued
Recall that \( \text{int}(x) \) is equal to the greatest integer less than or equal to \( x \). For example, \( \text{int}(3) = 3 \). From the definition of \( f \) and its graph in Figure 10.14 we can see that

\[
\begin{align*}
(a) & \quad \lim_{x \to 3^-} \text{int}(x) = 2 \\
(b) & \quad \lim_{x \to 3^+} \text{int}(x) = 3 \\
(c) & \quad \lim_{x \to 3} \text{int}(x) \text{ does not exist.}
\end{align*}
\]

Now try Exercise 41.

**SOLUTION**

**Limits Involving Infinity**

The informal definition that we have for a limit refers to \( \lim_{x \to a} f(x) = L \) where both \( a \) and \( L \) are real numbers. In Section 10.2 we adapted the definition to apply to limits of the form \( \lim_{x \to \infty} f(x) = L \) so that we could use this notation in describing definite integrals. This is one type of “limit at infinity.” Notice that the limit itself \( L \) is a finite real number, assuming the limit exists, but that the values of \( x \) are approaching infinity.

**NOTES ON EXAMPLES**

Examples 5 and 6 demonstrate that it is not necessarily true that \( \lim_{x \to a} f(x) = f(c) \). Make certain that students understand the role of continuity in evaluating limits.

**INFINITE LIMITS ARE NOT LIMITS**

It is important to realize that an infinite limit is not a limit, despite what the name might imply. It describes a special case of a limit that does not exist. Recall that a sawhorse is not a horse and a badminton bird is not a bird.

**ARCHIMEDES (ca. 287–212 B.C.)**

The Greek mathematician Archimedes found the area of a circle using a method involving infinite limits. See Exercise 89 for a modern version of his method.

Notice that limits, whether at \( a \) or at infinity, are always finite real numbers; otherwise, the limits do not exist. For example, it is correct to write

\[
\lim_{x \to 0} \frac{1}{x^2} \text{ does not exist,}
\]

since it approaches no real number \( L \). In this case, however, it is also convenient to write

\[
\lim_{x \to 0} \frac{1}{x^2} = \infty,
\]

which gives us a little more information about why the limit fails to exist. (It increases without bound.) Similarly, it is convenient to write

\[
\lim_{x \to 0^+} \ln x = -\infty,
\]

since \( \ln x \) decreases without bound as \( x \) approaches 0 from the right. In this context, the symbols “\( \infty \)” and “\( -\infty \)” are sometimes called infinite limits.
EXAMPLE 7 Investigating Limits as \( x \to \pm \infty \)

Let \( f(x) = \frac{\sin x}{x} \). Find \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

**SOLUTION** The graph of \( f \) in Figure 10.15 suggests that

\[
\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to -\infty} \frac{\sin x}{x} = 0.
\]

*Now try Exercise 47.*

In Section 1.3, we used limits to describe the *unbounded behavior* of the function \( f(x) = x^3 \) as \( x \to \pm \infty \):

\[
\lim_{x \to \infty} x^3 = \infty \quad \text{and} \quad \lim_{x \to -\infty} x^3 = -\infty.
\]

The behavior of the function \( g(x) = e^x \) as \( x \to \pm \infty \) can be described by the following two limits:

\[
\lim_{x \to \infty} e^x = \infty \quad \text{and} \quad \lim_{x \to -\infty} e^x = 0.
\]

The function \( g(x) = e^x \) has unbounded behavior as \( x \to \infty \) and has a finite limit as \( x \to -\infty \).

EXAMPLE 8 Using Tables to Investigate Limits as \( x \to \pm \infty \)

Let \( f(x) = xe^{-x} \). Find \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

**SOLUTION** The tables in Figure 10.16 suggest that

\[
\lim_{x \to \infty} xe^{-x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} xe^{-x} = -\infty.
\]

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<td>1E-20</td>
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<tr>
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<td>5E-25</td>
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\( Y_2 = Xe^{(-X)} \)

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<th>( Y )</th>
</tr>
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<tbody>
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\( Y_2 = Xe^{(-X)} \)

*FIGURE 10.16* The table in (a) suggests that the values of \( f(x) = xe^{-x} \) approach 0 as \( x \to \infty \) and the table in (b) suggests that the values of \( f(x) = xe^{-x} \) approach \(-\infty\) as \( x \to -\infty \). (Example 8)

The graph of \( f \) in Figure 10.17 supports these results.

*Now try Exercise 49.*
In Section 2.6 we used the graph of \( f(x) = \frac{1}{x-2} \) to state that
\[
\lim_{x \to 2^-} \frac{1}{x-2} = -\infty \quad \text{and} \quad \lim_{x \to 2^+} \frac{1}{x-2} = \infty.
\]
Either one of these unbounded limits allows us to conclude that the vertical line \( x = 2 \) is a vertical asymptote of the graph of \( f \) (Figure 10.18).

**EXAMPLE 9 Investigating Unbounded Limits**

Find \( \lim_{x \to 2} \frac{1}{x-2} \).

**SOLUTION** The graph of \( f(x) = \frac{1}{x-2} \) in Figure 10.19 suggests that
\[
\lim_{x \to 2^-} \frac{1}{x-2} = \infty \quad \text{and} \quad \lim_{x \to 2^+} \frac{1}{x-2} = \infty.
\]
This means that the limit of \( f \) as \( x \) approaches 2 does not exist. The table of values in Figure 10.20 agrees with this conclusion. The graph of \( f \) has a vertical asymptote at \( x = 2 \).

Not all zeros of denominators correspond to vertical asymptotes as illustrated in Examples 5 and 7.

**EXAMPLE 10 Investigating a Limit at \( x = 0 \)**

Find \( \lim_{x \to 0} \frac{\sin x}{x} \).

**SOLUTION** The graph of \( f(x) = \frac{\sin x}{x} \) in Figure 10.15 suggests this limit exists. The table of values in Figure 10.21 suggest that
\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]
QUICK REVIEW 10.3
(For help, go to Sections 1.2 and 1.3.)

In Exercises 1 and 2, find (a) \( f(-2) \), (b) \( f(0) \), and (c) \( f(2) \).

1. \( f(x) = \frac{2x + 1}{(2x - 4)^2} \)

2. \( f(x) = \frac{\sin x}{x} \)

In Exercises 3 and 4, find the (a) vertical asymptotes and (b) horizontal asymptotes of the graph of \( f \), if any.

3. \( f(x) = \frac{2x^2 + 3}{x^2 - 4} \)

4. \( f(x) = \frac{x^3 + 1}{2 - x - x^2} \)

In Exercises 5 and 6, the end behavior asymptote of the function \( f \) is one of the following. Which one is it?

(a) \( y = 2x^2 \)  
(b) \( y = -2x^2 \)  
(c) \( y = x^3 \)  
(d) \( y = -x^3 \)

5. \( f(x) = \frac{2x^3 - 3x^2 + 1}{3 - x} \)

6. \( f(x) = \frac{x^4 + 2x^2 + x + 1}{x - 3} \)

In Exercises 7 and 8, find (a) the points of continuity and (b) the points of discontinuity of the function.

7. \( f(x) = \sqrt{x + 2} \)

8. \( g(x) = \frac{2x + 1}{x^2 - 4} \)

Exercises 9 and 10 refer to the piecewise-defined function

\[ f(x) = \begin{cases} 3x + 1 & \text{if } x \leq 1 \\ 4 - x^2 & \text{if } x > 1 \end{cases} \]

9. Draw the graph of \( f \).

10. Find the points of continuity and the points of discontinuity of \( f \).

SECTION 10.3 EXERCISES

In Exercises 1–10, find the limit by direct substitution if it exists.

1. \( \lim_{x \to -3} \frac{x(x - 1)^2 - 4}{x^2 - 9} \)

2. \( \lim_{x \to 2} (x^3 - 2x + 3) \)

3. \( \lim_{x \to 0} (e^x \sin x) \)

4. \( \lim_{x \to -2} \frac{x^3 + 1}{x + 1} \)

5. \( \lim_{x \to -2} \frac{x^2 - 4}{x + 2} \)

6. \( \lim_{x \to 0} \sqrt{x - 3} \)

7. \( \lim_{x \to -1} \frac{x^2 + 7x + 12}{x^2 - 9} \)

8. \( \lim_{x \to -1} \frac{x^2 - 9}{x^2 + 2x - 15} \)

9. \( \lim_{x \to 0} \frac{x^2 - 1}{x^2 + 1} \)

10. \( \lim_{x \to 0} \frac{a^2 - 1}{a^2 + 1} \)

In Exercises 11–18, (a) explain why you cannot use substitution to find the limit and (b) find the limit algebraically if it exists.

11. \( \lim_{x \to -3} \frac{x^3 + 1}{x^2 - 9} \)

12. \( \lim_{x \to 3} \frac{x^3 - 2x^2 + 2}{x^2 - 1} \)

13. \( \lim_{x \to -1} \frac{|x^2 - 4|}{x + 2} \)

14. \( \lim_{x \to 2} \frac{|x^2 - 4|}{x + 2} \)

15. \( \lim_{x \to 0} \frac{x - 2}{x^2} \)

16. \( \lim_{x \to 0} \frac{x}{x^2} \)

17. \( \lim_{x \to 0} \frac{\sin x}{2x^2 - x} \)

18. \( \lim_{x \to 0} \frac{\sin x}{x^2} \)

In Exercises 19–22, use the fact that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), along with the limit properties, to find the following limits.

19. \( \lim_{x \to 0} \frac{\sin x}{2x^2 - x} \)

20. \( \lim_{x \to 0} \frac{\sin 3x}{x} \)

21. \( \lim_{x \to 0} \frac{\sin x}{x^2} \)

22. \( \lim_{x \to 0} \frac{x + \sin x}{2x} \)

23. \( \lim_{x \to 0} \frac{e^x - \sqrt{x}}{x} \)

24. \( \lim_{x \to 0} \frac{3\sin x - 4\cos x}{\sin x + \cos x} \)

25. \( \lim_{x \to \pi/2} \frac{\ln(x \sin x)}{\sin x} \)

26. \( \lim_{x \to \pi/2} \frac{\sqrt{x + 9} - \ln x}{\log_3 x} \)

In Exercises 27–30, use the given graph to find the limits or to explain why the limits do not exist.

27. \( \lim_{x \to 2} f(x) \)

28. \( \lim_{x \to 3} f(x) \)

29. \( \lim_{x \to 3} f(x) \)

30. \( \lim_{x \to 3} f(x) \)
30. (a) \( \lim_{x \to 1} f(x) = 1 \)
(b) \( \lim_{x \to 1} f(x) = 3 \)
(c) \( \lim_{x \to 1} f(x) \)

In Exercises 31 and 32, the graph of a function \( y = f(x) \) is given. Which of the statements about the function are true and which are false?

31. (a) \( \lim_{x \to 1} f(x) = 1 \) true
(b) \( \lim_{x \to 0^+} f(x) = 0 \) true
(c) \( \lim_{x \to 1^-} f(x) = 1 \) false
(d) \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \)
(e) \( \lim_{x \to 0} f(x) \) exists false
(f) \( \lim_{x \to 0^+} f(x) = 0 \) false
(g) \( \lim_{x \to 0} f(x) = 1 \) false
(h) \( \lim_{x \to 1^-} f(x) = 1 \) true
(i) \( \lim_{x \to 1^-} f(x) = 0 \) false
(j) \( \lim_{x \to 0} f(x) = 2 \) true

32. (a) \( \lim_{x \to 1} f(x) = 1 \) true
(b) \( \lim_{x \to 2^+} f(x) \) does not exist
(c) \( \lim_{x \to 2^-} f(x) = 2 \) false
(d) \( \lim_{x \to 1^+} f(x) = 2 \) true
(e) \( \lim_{x \to 1^-} f(x) = 1 \)
(f) \( \lim_{x \to 1} f(x) \) does not exist
(g) \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \) false
(h) \( \lim_{x \to 0} f(x) \) exists for every \( c \) in \((-1, 1)\). False (not \( x = 0 \))
(i) \( \lim_{x \to 1} f(x) \) exists for every \( c \) in \((1, 3)\). True

In Exercises 33 and 34, use a graph of \( f \) to find (a) \( \lim_{x \to a} f(x) \), (b) \( \lim_{x \to a} f(x) \) and (c) \( \lim_{x \to a} f(x) \) if they exist.

33. \( f(x) = (1 + x)^{1/x} \)
34. \( f(x) = (1 + x)^{1/(2x)} \)

35. **Group Activity** Assume that \( \lim_{x \to 1} f(x) = -1 \) and \( \lim_{x \to 1} g(x) = 4 \). Find the limit.
(a) \( \lim_{x \to 1} (g(x) + 2) = 6 \)
(b) \( \lim_{x \to 1} xf(x) = -4 \)
(c) \( \lim_{x \to 1} g^2(x) = 16 \)
(d) \( \lim_{x \to 1} g(x) - f(x) = -2 \)

36. **Group Activity** Assume that \( \lim_{x \to a} f(x) = 2 \) and \( \lim_{x \to a} g(x) = -3 \). Find the limit.
(a) \( \lim_{x \to a} (f(x) + g(x)) = -1 \)
(b) \( \lim_{x \to a} (f(x) \cdot g(x)) = -6 \)
(c) \( \lim_{x \to a} (3g(x) + 1) = -8 \)
(d) \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{2}{3} \)

In Exercises 37–40, complete the following for the given piecewise-defined function \( f \).
(a) Draw the graph of \( f \).
(b) Determine \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \).
(c) **Writing to Learn** Does \( \lim_{x \to a} f(x) \) exist? If it does, give its value. If it does not exist, give an explanation.

37. \( a = 2, f(x) = \begin{cases} 2 - x & \text{if } x < 2 \\ 1 & \text{if } x = 2 \\ x^2 - 4 & \text{if } x > 2 \end{cases} \)
38. \( a = 1, f(x) = \begin{cases} 2 - x & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases} \)
39. \( a = 0, f(x) = \begin{cases} |x - 3| & \text{if } x < 0 \\ x^2 - 2x & \text{if } x \geq 0 \end{cases} \)
40. \( a = -3, f(x) = \begin{cases} 1 - x^2 & \text{if } x \geq -3 \\ 8 - x & \text{if } x < -3 \end{cases} \)

In Exercises 41–46, find the limit.
41. \( \lim_{x \to 2^+} \int x \, dx = 2 \)
42. \( \lim_{x \to 2^-} \int x \, dx = 1 \)
43. \( \lim_{x \to 0^{0.001}} \int (x + 1) \, dx = 0 \)
44. \( \lim_{x \to \infty} \int (x^2 + 1) \, dx = 4 \)
45. \( \lim_{x \to -3^+} \left| \frac{x + 3}{x} \right| = 1 \)
46. \( \lim_{x \to \infty} \frac{5x}{2x} = 2 \)

In Exercises 47–54, find (a) \( \lim_{x \to \infty} y \) and (b) \( \lim_{x \to \infty} y \).
47. \( y = \frac{x}{x + 1} \) \( y = 0 \)
48. \( y = \frac{x + \sin x}{x} \) \( y = 1 \)
49. \( y = \frac{x}{1 + 2x} \) \( y = 1 \)
50. \( y = \frac{x}{2x} \) \( y = 0 \)
51. \( y = x + \sin x \)
52. \( y = e^{-x} + \sin x \)
53. \( y = -e^x \sin x \)
54. \( y = e^{-x} \cos x \)

In Exercises 55–60, use graphs and tables to find the limit and identify any vertical asymptotes.
55. \( \lim_{x \to 3} \frac{1}{x - 3} = -\infty; x = 3 \)
56. \( \lim_{x \to 0^+} \frac{x}{x - 3} = \infty; x = 3 \)
57. \( \lim_{x \to \infty} \frac{x + 2}{x} = \infty; x = -2 \)
58. \( \lim_{x \to \infty} \frac{x}{x + 2} = \infty; x = -2 \)
59. \( \lim_{x \to 5} \frac{1}{(x - 5)^2} = \infty; x = 5 \)
60. \( \lim_{x \to 2} \frac{1}{x^2 - 4} = -\infty; x = 2 \)

In Exercises 61–64, determine the limit algebraically if possible. Support your answer graphically.
61. \( \lim_{x \to \infty} \frac{(1 + x)^3 - 1}{x} = 3 \)
62. \( \lim_{x \to \infty} \frac{1/(3 + x) - 1/3}{x} = -\frac{1}{9} \)
63. \( \lim_{x \to \infty} \frac{\tan x}{x} = 1 \)
64. \( \lim_{x \to \infty} \frac{x - 4}{x^3 - 4} = 0 \) none
In Exercises 65–72, find the limit.

65. \( \lim_{x \to 0} \frac{|x|}{x^2} = \) 0
66. \( \lim_{x \to 0} \frac{x^2}{|x|} = 0 \)
67. \( \lim_{x \to 0} \left[ x \sin \left( \frac{1}{x} \right) \right] = 0 \)
68. \( \lim_{x \to 27} \cos \left( \frac{1}{x} \right) = 0 \)
69. \( \lim_{x \to 1} \frac{x^2 + 1}{x - 1} = \) undefined
70. \( \lim_{x \to \infty} \ln \frac{x^2}{x} = 2 \)
71. \( \lim_{x \to \infty} \frac{\ln x}{x} = 1 \)
72. \( \lim_{x \to \infty} 3^x = 0 \)

**Standardized Test Questions**

73. **True or False** If \( f(x) = \begin{cases} x + 2 & \text{if } x \leq 3 \\ 8 - x & \text{if } x > 3 \end{cases} \), then \( \lim_{x \to 3} f(x) \) is undefined. Justify your answer.
74. **True or False** If \( f(x) \) and \( g(x) \) are two functions and \( \lim_{x \to 0} f(x) \) does not exist, then \( \lim_{x \to 0} [f(x) \cdot g(x)] \) cannot exist. Justify your answer.

**Multiple Choice** In Exercises 75–78, match the function \( y = f(x) \) with the table. Do not use a calculator.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.7</td>
<td>5.7</td>
</tr>
<tr>
<td>2.8</td>
<td>5.9</td>
</tr>
<tr>
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<td>6.2</td>
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<td>3.3</td>
<td>6.8</td>
</tr>
<tr>
<td>( x=2.7 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
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<th>( Y )</th>
</tr>
</thead>
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<tr>
<td>2.8</td>
<td>33.8</td>
</tr>
<tr>
<td>3</td>
<td>63.9</td>
</tr>
<tr>
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<td>-55.9</td>
</tr>
<tr>
<td>3.2</td>
<td>-26.9</td>
</tr>
<tr>
<td>3.3</td>
<td>-56.7</td>
</tr>
<tr>
<td>( x=2.7 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>( Y )</th>
</tr>
</thead>
<tbody>
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<td>98.9</td>
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<tr>
<td>3</td>
<td>194.9</td>
</tr>
<tr>
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<td>290.9</td>
</tr>
<tr>
<td>3.2</td>
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<td>3.3</td>
<td>482.9</td>
</tr>
<tr>
<td>( x=2.7 )</td>
<td></td>
</tr>
</tbody>
</table>

75. \( y = \frac{x^2 - 2x - 3}{x - 3} \) \( B \)
76. \( y = \frac{x^2 + 2x + 3}{x - 3} \) \( A \)
77. \( y = \frac{x^2 - 2x - 9}{x - 3} \) \( C \)
78. \( y = \frac{x^3 - 27}{x - 3} \) \( D \)

**Explorations**

In Exercises 79–82, complete the following for the given piecewise-defined function \( f(x) \).

(a) Draw the graph of \( f \).
(b) At what points \( c \) in the domain of \( f \) does \( \lim f(x) \) exist?
(c) At what points \( c \) does only the left-hand limit exist?
(d) At what points \( c \) does only the right-hand limit exist?

79. \( f(x) = \begin{cases} \cos x & \text{if } -\pi \leq x < 0 \\ -\cos x & \text{if } 0 \leq x \leq \pi \end{cases} \)
80. \( f(x) = \begin{cases} \sin x & \text{if } -\pi \leq x < 0 \\ \csc x & \text{if } 0 \leq x \leq \pi \end{cases} \)
81. \( f(x) = \begin{cases} \sqrt{1 - x^2} & \text{if } -1 \leq x < 0 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } 0 \leq x < 1 \end{cases} \)
82. \( f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } x < -2 \text{ or } x > 2 \end{cases} \)

**Rabbit Population** The population of rabbits over a 2-year period in a certain county is given in Table 10.5.

<table>
<thead>
<tr>
<th>Month (in thousands)</th>
<th>Number (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
</tr>
<tr>
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<td>48</td>
</tr>
<tr>
<td>20</td>
<td>50</td>
</tr>
<tr>
<td>22</td>
<td>51</td>
</tr>
</tbody>
</table>

(a) Draw a scatter plot of the data in Table 10.5.
(b) Find a logistic regression model for the data. Find the limit of that model as time approaches infinity.
(c) What can you conclude about the limit of the rabbit population growth in the county?
(d) Provide a reasonable explanation for the population growth limit.
**Group Activity** In Exercises 84–87, sketch a graph of a function \( y = f(x) \) that satisfies the stated conditions. Include any asymptotes.

84. \( \lim_{x \to 0} f(x) = \infty, \lim_{x \to \infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = 2 \)

85. \( \lim_{x \to 0} f(x) = -\infty, \lim_{x \to \infty} f(x) = -\infty, \lim_{x \to -\infty} f(x) = 2 \)

86. \( \lim_{x \to 1} f(x) = 2, \lim_{x \to 2-} f(x) = -\infty, \lim_{x \to 2+} f(x) = \infty \)

87. \( \lim_{x \to 1} f(x) = \infty, \lim_{x \to 2-} f(x) = -\infty, \lim_{x \to 2+} f(x) = -\infty, \lim_{x \to -\infty} f(x) = \infty \)

**Extending the Ideas**

88. **Properties of Limits** Find the limits of \( f, g \), and \( fg \) as \( x \) approaches \( c \).

(a) \( f(x) = \frac{2}{x^2}, g(x) = x^2, c = 0 \implies \frac{2}{0} = 2 \)

(b) \( f(x) = \frac{1}{x^3}, g(x) = \sqrt{x}, c = 0 \implies \frac{1}{0^3} \text{ is undefined} \)

(c) \( f(x) = \frac{3}{x - 1}, g(x) = (x - 1)^2, c = 1 \implies \frac{3}{0} = 0 \)

(d) \( f(x) = \frac{1}{(x - 1)^4}, g(x) = (x - 1)^2, c = 1 \implies \frac{1}{0} = 0 \)

(e) **Writing to Learn** Suppose that \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} g(x) = 0 \). Based on your results in parts (a)–(d), what can you say about \( \lim_{x \to \infty} (f(x) \cdot g(x)) \)?

**Note:** nothing

89. **Limits and the Area of a Circle** Consider an \( n \)-sided regular polygon made up of \( n \) congruent isosceles triangles, each with height \( h \) and base \( b \). The figure shows an 8-sided regular polygon.

- **(a)** Show that the area of an 8-sided regular polygon is \( A = 4bh \) and the area of the \( n \)-sided regular polygon is \( A = (1/2)nbh \).
- **(b)** Show that the base \( b \) of the \( n \)-sided regular polygon is \( b = 2h \tan \left( \frac{180^\circ}{n} \right) \).
- **(c)** Show that the area \( A \) of the \( n \)-sided regular polygon is \( A = nh^2 \tan \left( \frac{180^\circ}{n} \right) \).
- **(d)** Let \( h = 1 \). Construct a table of values for \( n \) and \( A \) for \( n = 4, 8, 16, 100, 500, 1000, 5000, 10000, 100000 \). Does \( A \) have a limit as \( n \to \infty \)?
- **(e)** Repeat part (d) with \( h = 3 \).
- **(f)** Give a convincing argument that \( \lim_{n \to \infty} = \pi h^2 \), the area of a circle of radius \( h \).

90. **Continuous Extension of a Function** Let

\[
\begin{aligned}
f(x) &= \begin{cases} 
    x^2 - 3x + 3 & \text{if } x \neq 2 \\
    a & \text{if } x = 2 
\end{cases}
\end{aligned}
\]

(a) Sketch several possible graphs for \( f \).

(b) Find a value for \( a \) so that the function is continuous at \( x = 2 \).

In Exercises 91–93, (a) graph the function, (b) verify that the function has one removable discontinuity, and (c) give a formula for a continuous extension of the function. [Hint: See Exercise 90.]

91. \( y = \frac{2x + 4}{x + 2} \)

92. \( y = \frac{x - 5}{5 - x} \)

93. \( y = \frac{x^3 - 1}{x - 1} \)

(a) \( y = \frac{2x + 4}{x + 2} = \frac{2(x + 2)}{x + 2} = 2 \)

(b) \( y = \frac{x - 5}{5 - x} = \frac{-|x - 5|}{x - 5} = -1 \)

(c) \( y = -1 \)

92. \( y = \frac{x^3 - 1}{x - 1} = \frac{(x - 1)(x^2 + x + 1)}{x - 1} = x^2 + x + 1 \)

(c) \( y = x^2 + x + 1 \)
10.4 Numerical Derivatives and Integrals

What you’ll learn about
- Derivatives on a Calculator
- Definite Integrals on a Calculator
- Computing a Derivative from Data
- Computing a Definite Integral from Data

...and why
The numerical capabilities of a graphing calculator make it easy to perform many calculations that would have been exceedingly difficult in the past.

Derivatives on a Calculator

As computers and sophisticated calculators have become indispensable tools for modern engineers and mathematicians (and, ultimately, for modern students of mathematics), numerical techniques of differentiation and integration have re-emerged as primary methods of problem-solving. This is no small irony, as it was precisely to avoid the tedious computations inherent in such methods that calculus was invented in the first place. Although nothing can diminish the magnitude of calculus as a significant human achievement, and although nobody can get far in mathematics or science without it, the modern fact is that applying the old-fashioned methods of limiting approximations—with the help of a calculator—is often the most efficient way to solve a calculus problem.

Most graphing calculators have built-in algorithms that will approximate derivatives of functions numerically with good accuracy at most points of their domains. We will use the notation NDER \( f \) to denote such a calculator derivative approximation to \( f \).

For small values of \( h \), the regular difference quotient

\[
\frac{f(a + h) - f(a)}{h}
\]

is often a good approximation of \( f'(a) \). However, the same value of \( h \) will usually produce a better approximation of \( f'(a) \) if we use the symmetric difference quotient

\[
\frac{f(a + h) - f(a - h)}{2h},
\]

as illustrated in Figure 10.22.

Many graphing utilities use the symmetric difference quotient with a default value of \( h = 0.001 \) for computing NDER \( f \). When we refer to the numerical derivative in this book, we will assume that it is the symmetric difference quotient with \( h = 0.001 \).

**DEFINITION** Numerical Derivative

In this book, we define the **numerical derivative of \( f \) at \( a \)** to be

\[
NDER f(a) = \frac{f(a + 0.001) - f(a - 0.001)}{0.002}.
\]

Similarly, we define the **numerical derivative of \( f \)** to be the function

\[
NDER f(x) = \frac{f(x + 0.001) - f(x - 0.001)}{0.002}.
\]
EXAMPLE 1 Computing a Numerical Derivative

Let \( f(x) = x^3 \). Compute \( \text{NDER} f(2) \) by calculating the symmetric difference quotient with \( h = 0.001 \). Compare it to the actual value of \( f''(x) \).

**SOLUTION**

\[
\text{NDER} f(x) = \frac{f(2 + 0.001) - f(2 - 0.001)}{0.002} = \frac{f(2.001) - f(1.999)}{0.002} = 12.000001
\]

The actual value is

\[
f''(2) = \lim_{h \to 0} \frac{(2 + h)^3 - 2^3}{h} = \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h \to 0} (12 + 6h + h^2) = 12
\]

The numerical derivative in this case is obviously quite accurate. In practice, it is not necessary to key in the symmetric difference quotient, as it is done by the calculator with its built-in algorithm. Figure 10.23 shows the command that would be used on one such calculator to find the numerical derivative in Example 1.

If \( f'(a) \) exists, then \( \text{NDER} f(a) \) usually gives a good approximation to the actual value. On the other hand, the algorithm will sometimes return a value for \( \text{NDER} f(a) \) when \( f'(a) \) does not exist. (See Exercise 51.)

**Definite Integrals on a Calculator**

Recall from the history of the area problem (Section 10.2) that the strategy of summing up thin rectangles to approximate areas is ancient. The thinner the rectangles, the better the approximation—and, of course, the more tedious the computation. Today, thanks to technology, we can employ the ancient strategy without the tedium.

Many graphing calculators have built-in algorithms to compute definite integrals with great accuracy. We use the notation \( \text{NINT} (f(x), x, a, b) \) to denote such a calculator approximation to \( \int_{a}^{b} f(x) \, dx \). Unlike NDER, which uses a fixed value of \( \triangle x \), NINT will vary the value of \( \triangle x \) until the numerical integral gets close to a limiting value, often resulting in an exact answer (at least to the number of digits in the calculator display). Because the algorithm for NINT finds the definite integral by Riemann sum approximation rather than by calculus, we call it a **numerical integral**.
EXAMPLE 2  Finding a Numerical Integral

Use NINT to find the area of the region $R$ enclosed between the $x$-axis and the graph of $y = 1/x$ from $x = 1$ to $x = 4$.

**SOLUTION**  The region is shown in Figure 10.24.

The area can be written as the definite integral $\int_{1}^{4} \frac{1}{x} \, dx$, which we find on a graphing calculator: $\text{NINT}(1/x, 1, 4) = 1.386294361$. The exact answer (as you will learn in a calculus course) is $\ln 4$, which agrees in every displayed digit with the NINT value!

Figure 10.25 shows the syntax for numerical integration on one type of calculator.

*Now try Exercise 13.*

---

EXPLORATION 1  A Do-It-Yourself Numerical Integrator

Recall that a definite integral is the limit at infinity of a Riemann sum—that is, a sum of the form $\sum_{k=1}^{n} f(x_k) \Delta x$. You can use your calculator to evaluate sums of sequences using LIST commands. (It is not as accurate as NINT, and certainly not as easy, but at least you can see the summing that takes place.)

1. The integral in Example 2 can be computed using the command $\text{sum(seq(1/(1 + K \cdot 3/50) \cdot 3/50, K, 1, 50))}$.
   This uses 50 RRAM rectangles, each with width $\Delta x = 3/50$. Find the sum on your calculator and compare it to the NINT value.

2. Study the command until you see how it works. Adapt the command to find the RRAM approximation for 100 rectangles and compute it on your calculator. Does the approximation get better?

3. What definite integral is approximated by the command $\text{sum(seq(sin(0 + K \cdot \pi/50) \cdot \pi/50, K, 1, 50))}$?
   Compute it on your calculator and compare it to the NINT value for the same integral.

4. Write a command that uses 50 RRAM rectangles to approximate $\int_{4}^{9} \sqrt{x} \, dx$. Compute it on your calculator and compare it to the NINT value for the same integral.

---

EXPLORATION EXTENSIONS

Which of the following would you expect typically to give the most accurate estimate of the area under a curve using $n$ rectangles: the LRAM approximation, the RRAM approximation, or the average of these two?

---

Remember that we were originally motivated to find areas because of their connection to the problem of distance traveled. To show just one of the many applications of integration, we use the numerical integral to solve a distance problem in Example 3.

---

EXAMPLE 3  Finding Distance Traveled

An automobile is driven at a variable rate along a test track for 2 hours so that its velocity at any time $t \ (0 \leq t \leq 2)$ is given by $v(t) = 30 + 10 \sin 6t$ miles per hour. How far does the automobile travel during the 2-hour test?

*continued*
**SOLUTION** According to the analysis found in Section 10.2, the distance traveled is given by \[ \int_0^2 (30 + 10 \sin (6t)) \, dt. \] We use a calculator to find the numerical integral:

\[ \text{NINT} (30 + 10 \sin (6t), t, 0, 2) \approx 60.26. \]

Interpreting the answer, we conclude that the automobile travels 60.26 miles.

### Computing a Derivative from Data

Sometimes all we are given about a problem situation is a scatter plot obtained from a set of data—a numerical model of the problem. There are two ways to get information about the derivative of the model.

1. To approximate the derivative at a point: Remember that the average rate of change over a small interval, \( \frac{\Delta y}{\Delta x} \), approximates the derivative at points in that interval. (Generally, the approximation is better near the middle of the interval than it is near the endpoints.) The average rate of change on an interval between two data points can be computed directly from the data.

2. To approximate the derivative function: Regression techniques can be used to fit a curve to the data, and then NDER can be applied to the regression model to approximate the derivative. Alternatively, the values of \( \frac{\Delta y}{\Delta x} \) can be plotted, and then regression techniques can be used to fit a derivative approximation through those points.

#### EXAMPLE 4 Finding Derivatives from Data

Table 10.6 shows the height (in feet) of a falling ball above ground level as measured by a motion detector at time intervals of 0.04 seconds.

(a) Estimate the instantaneous speed of the ball at \( t = 0.2 \) seconds.

(b) Draw a scatter plot of the data and use quadratic regression to model the height \( s \) of the ball above the ground as a function of \( t \).

(c) Use NDER to approximate \( s'(0.2) \) and compare it to the value found in (a).

**SOLUTION**

(a) Since 0.2 is the midpoint of the time interval \([0.16, 0.24]\), the average rate of change \( \frac{\Delta s}{\Delta t} \) on the interval \([0.16, 0.24]\) should give a good approximation to \( s'(0.2) \):

\[ s'(0.2) = \frac{\Delta s}{\Delta t} = \frac{4.30 - 5.45}{0.24 - 0.16} = -14.375. \]

The speed is about 14.375 feet per second at \( t = 0.2 \).

(b) The scatter plot is shown in Figure 10.26, along with the quadratic regression curve. A graphing calculator gives \( s(t) \approx -17.12t^2 - 7.74t + 7.13 \) as the equation of the quadratic regression model.

(c) The calculator computes NDER \( s(0.2) \) to be \(-14.588\), which agrees quite well with the approximation in (a). In fact, the difference is only 0.213 feet per second, less than 1.5% of the speed of the ball.
EXAMPLE 5 Finding Derivatives from Data

This example also uses the falling ball data in Table 10.6.

(a) Compute the average velocity, \( \Delta s/\Delta t \), on each subinterval of length 0.04. Make a table showing the midpoints of the subintervals in one column and the values of \( \Delta s/\Delta t \) in the second column.

(b) Make a scatter plot showing the numbers in the second column as a function of the numbers in the first column and find a linear regression model to model the data.

(c) Use the linear regression model in (b) to approximate the velocity of the ball at \( t = 0.2 \) and compare it to the values found in Example 4.

SOLUTION

(a) The first subinterval, which begins at 0.04 and ends at 0.08, has a midpoint of 0.06. On that interval, \( \Delta s/\Delta t = (6.40 - 6.80)/(0.08 - 0.04) = -10.00 \). The rest of the midpoints and values of \( \Delta s/\Delta t \) are computed similarly and are shown in Table 10.7.

(b) The scatter plot and the regression line are shown in Figure 10.27. A graphing calculator gives \( v(t) = -34.470t - 7.727 \) as the regression line.

(c) The linear regression model gives \( v(0.2) \approx -14.62 \), which is close to the values found in Example 4.

Computing a Definite Integral from Data

If we are given a set of data points, the \( x \)-coordinates of the points define subintervals between the smallest and largest \( x \)-values in the data. We can form a Riemann sum

\[
\sum_{k=1}^{n} f(x_k) \Delta x
\]

using the lengths of the subintervals for \( \Delta x \) and either the left or right endpoints of the intervals as the \( x_k \)'s. The Riemann sum then approximates the definite integral of the function over the interval.

Example 6 illustrates how this is done.
Table 10.8 Velocity of the Moving Body

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Velocity (m/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.25</td>
<td>0.28</td>
</tr>
<tr>
<td>0.50</td>
<td>0.53</td>
</tr>
<tr>
<td>0.75</td>
<td>0.73</td>
</tr>
<tr>
<td>1.00</td>
<td>0.90</td>
</tr>
<tr>
<td>1.25</td>
<td>1.01</td>
</tr>
<tr>
<td>1.50</td>
<td>1.11</td>
</tr>
<tr>
<td>1.75</td>
<td>1.18</td>
</tr>
<tr>
<td>2.00</td>
<td>1.21</td>
</tr>
<tr>
<td>2.25</td>
<td>1.17</td>
</tr>
<tr>
<td>2.50</td>
<td>1.13</td>
</tr>
<tr>
<td>2.75</td>
<td>1.05</td>
</tr>
<tr>
<td>3.00</td>
<td>0.91</td>
</tr>
<tr>
<td>3.25</td>
<td>0.72</td>
</tr>
<tr>
<td>3.50</td>
<td>0.55</td>
</tr>
<tr>
<td>3.75</td>
<td>0.26</td>
</tr>
</tbody>
</table>

EXAMPLE 6  Finding a Definite Integral Using Data

Table 10.8 shows the velocity of a moving body (in meters per second) measured at regular quarter-second intervals. Estimate the distance traveled by the body from $t = 0$ to $t = 3.75$.

SOLUTION  Figure 10.28 gives a scatter plot of the velocity data.

The distance traveled is $\int_0^{3.75} v(t) \, dt$, which we approximate with a Riemann sum constructed directly from the data. We sum up 15 products of the form $v(t_k) \, \Delta t$, using the right endpoint for $t_k$ each time. (This is the RRAM approximation in the notation of Section 10.2.) Note that $\Delta t = 0.25$ for every subinterval.

$$\int_0^{3.75} v(t) \, dt = \sum_{k=1}^{15} v(t_k) \, \Delta t$$

$$= 0.25(0.28 + 0.53 + 0.73 + 0.90 + 1.01 + 1.11$$
$$+ 1.18 + 1.21 + 1.17 + 1.13 + 1.05 + 0.91 + 0.72$$
$$+ 0.55 + 0.26)$$

$$= 3.185$$

So the distance traveled by the body is about 3.2 meters.

Now try Exercise 27.
In Exercises 1–10, use NDER on a calculator to find the numerical derivative of the function at the specified point.

1. \( f(x) = 1 - x^2 \) at \( x = 2 \)
2. \( f(x) = 2x + \frac{1}{2} x^2 \) at \( x = 2 \)
3. \( f(x) = 3x^2 + 2 \) at \( x = -2 \)
4. \( f(x) = x^2 - 3x + 1 \) at \( x = 1 \)
5. \( f(x) = |x + 2| \) at \( x = -2 \)
6. \( f(x) = \frac{1}{x^2} \) at \( x = -1 \)
7. \( f(x) = \ln 2x \) at \( x = 1 \)
8. \( f(x) = 2 \ln x \) at \( x = 1 \)
9. \( f(x) = 3 \sin x \) at \( x = \pi \)
10. \( f(x) = \sin 3x \) at \( x = \pi \)

In Exercises 11–20, use NINT on a calculator to find the numerical integral of the function over the specified interval.

11. \( f(x) = x^2, [0, 4] \)
12. \( f(x) = x^3, [-4, 0] \)
13. \( f(x) = \sin x, [0, \pi] \)
14. \( f(x) = \sin x, [\pi, 2\pi] \)
15. \( f(x) = \cos x, [0, \pi] \)
16. \( f(x) = |\cos x|, [0, \pi] \)
17. \( f(x) = 1/x, [1, e] \)
18. \( f(x) = 1/x, [e, 2e] \)
19. \( f(x) = \frac{2}{1 + x^2}, [0, 10^9] \)
20. \( f(x) = \sec^2 x - \tan^2 x, [0, 10] \)

21. **Travel Time** A truck is driven at a variable rate for 3 hours so that its velocity at any time \( t \) (0 \( \leq t \leq 3 \)) is given by \( v(t) = 35 - 12 \cos 4t \) miles per hour. How far does the truck travel during the 3 hours? Round your answer to the nearest hundredth. \( 106.61 \) mi

22. **Travel Time** A bicyclist rides for 90 minutes, and her velocity at any time \( t \) hours (0 \( \leq t \leq 1.5 \)) is given by \( v(t) = 12 - 8 \sin 5t \) miles per hour. How far does she travel during the 90 minutes? Round your answer to the nearest hundredth. \( 16.95 \) mi

23. **Finding Derivatives from Data** A ball is dropped from the roof of a 30-story building. The height in feet above the ground of the falling ball is measured at 1/2-second intervals and recorded in the table.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Height (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>500</td>
</tr>
<tr>
<td>0.5</td>
<td>495</td>
</tr>
<tr>
<td>1.0</td>
<td>485</td>
</tr>
<tr>
<td>1.5</td>
<td>465</td>
</tr>
<tr>
<td>2.0</td>
<td>435</td>
</tr>
<tr>
<td>2.5</td>
<td>400</td>
</tr>
<tr>
<td>3.0</td>
<td>355</td>
</tr>
<tr>
<td>3.5</td>
<td>305</td>
</tr>
<tr>
<td>4.0</td>
<td>245</td>
</tr>
<tr>
<td>4.5</td>
<td>175</td>
</tr>
<tr>
<td>5.0</td>
<td>100</td>
</tr>
<tr>
<td>5.5</td>
<td>15</td>
</tr>
</tbody>
</table>

(a) Use the average velocity on the interval \([1, 2]\) to estimate the velocity of the ball at \( t = 1.5 \) seconds. \(-50 \) ft/sec

(b) Draw a scatter plot of the data.

(c) Find a quadratic regression model for the data.

(d) Use NDER of the model in part (c) to estimate the velocity of the ball at \( t = 1.5 \) seconds. \(-47.88 \) ft/sec

(e) Use the model to estimate how fast the ball is going when it hits the ground. \(\approx 179.28 \) ft/sec

24. **Estimating Average Rate of Change from Data**

Table 10.9 gives the U.S. Gross Domestic Product Data in billions of dollars for the years 1990–2003.

<table>
<thead>
<tr>
<th>Year</th>
<th>Amount (billions of dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>5803.1</td>
</tr>
<tr>
<td>1995</td>
<td>7397.7</td>
</tr>
<tr>
<td>1997</td>
<td>8304.3</td>
</tr>
<tr>
<td>1998</td>
<td>8747.0</td>
</tr>
<tr>
<td>1999</td>
<td>9268.4</td>
</tr>
<tr>
<td>2000</td>
<td>9817.0</td>
</tr>
<tr>
<td>2001</td>
<td>10,100.8</td>
</tr>
<tr>
<td>2002</td>
<td>10,480.8</td>
</tr>
<tr>
<td>2003</td>
<td>10,987.9</td>
</tr>
</tbody>
</table>

(a) Find the average rate of change of the gross domestic product from 1997 to 1998 and then from 2001 to 2002.

(b) Find a quadratic regression model for the data in Table 10.9 and overlay its graph on a scatter plot of the data. Let \( x = 0 \) stand for 1990, \( x = 1 \) stand for 1991, and so forth.

(c) Use the model in part (b), and a calculator NDER computation, to estimate the rate of change of the gross domestic product in 1997 and in 2001.

(d) Writing to Learn Use the model in part (b) to predict the gross domestic product in 2007. Is this reasonable? Why or why not?

25. Estimating Velocity Refer to the data in Exercise 23.

(a) Compute the average velocity, \( \frac{\Delta y}{\Delta x} \), on each subinterval of length 0.5. Make a table showing the midpoints of the subintervals in one column and the average velocities in the second column.

(b) Make a scatter plot showing the numbers in the second column as a function of the numbers in the first column and find a linear regression model to model the data.

(c) Use the linear regression model in part (b) to approximate the velocity of the ball at \( t = 1.5 \) seconds, and compare your result to the value found in Exercise 23(d).

26. Approximating Rate of Change Refer to the data in Exercise 24.

(a) Compute the average rate of change, \( \frac{\Delta y}{\Delta x} \), on each subinterval. Make a table showing the midpoints of the subintervals in one column and the average rates of change in the second column.

(b) Make a scatter plot showing the numbers in the second column as a function of the numbers in the first column and find a linear regression model to model the data.

(c) Use the linear regression model in part (b) to approximate the rate of change in 1997 and in 2001, and compare your results to the values found in Exercise 24(c).

27. Estimating Distance A stone is dropped from a cliff and its velocity (in feet per second) at regular 0.5-second intervals is shown in Table 10.10. Estimate the distance that the stone travels from \( t = 0 \) to \( t = 2.5 \). 100 ft

28. Estimating Distance Table 10.11 shows the velocity of a moving object in meters per second, measured at regular 0.2-second intervals. Estimate the distance traveled by the body from \( t = 0 \) to \( t = 1.6 \).

<table>
<thead>
<tr>
<th>Table 10.11 Velocity of the Object</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (sec)</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
</tr>
<tr>
<td>0.4</td>
</tr>
<tr>
<td>0.6</td>
</tr>
<tr>
<td>0.8</td>
</tr>
<tr>
<td>1.0</td>
</tr>
<tr>
<td>1.2</td>
</tr>
<tr>
<td>1.4</td>
</tr>
<tr>
<td>1.6</td>
</tr>
</tbody>
</table>

29. Writing to Learn Analyze the following program, which produces an LRAM approximation for the function entered in Y1 in the calculator. Then write a short paragraph explaining how it works.

```
PROGRAM: LRAM
:Input "A",A
:Input "B",B
:Input "N",N
:sum(seq(((B–A)/N)Y1(A+K((B–A)/N)),K,0,N–1))→C
:Disp "AREA =",C
```

30. Writing to Learn Analyze the following program, which produces an RRAM approximation for the function entered in Y1 in the calculator. Then write a short paragraph explaining how it works.

```
PROGRAM: RRAM
:Input "A",A
:Input "B",B
:Input "N",N
:sum(seq(((B–A)/N)Y1(A+K((B–A)/N)),K,1,N))→C
:Disp "AREA =",C
```
In Exercises 31–42, complete the following for the indicated interval \([a, b]\).

(a) Verify the given function is nonnegative.

(b) Use a calculator to find the LRAM, RRAM, and average approximations for the area under the graph of the function from \(x = a\) to \(x = b\) with 10, 20, 50, and 100 approximating rectangles. (You may want to use the programs in Exercises 29 and 30.)

(c) Compare the average area estimate in part (b) using 100 approximations for the area under the graph of the function from \(a\) to \(b\) with \(10, 20, 50,\) and \(100\) approximating rectangles. (You may want to use the programs in Exercises 29 and 30.)

### Standardized Test Questions

43. **True or False** The numerical derivative algorithm NDER always uses the same value of \(\Delta x\) (or \(h\)) to complete its calculations. Justify your answer.

44. **True or False** The numerical integral algorithm always uses the same value of \(\Delta x\) to complete its calculations. Justify your answer.

In Exercises 45–48, choose the correct answer. Do not use a calculator.

45. **Multiple Choice Estimating Area Under a Curve** Which of the following will typically produce the most accurate estimate of an area under a curve? \(B\)

(A) NDER \(\quad\) (B) NINT

(C) LRAM, 10 rectangles \(\quad\) (D) RRAM, 25 rectangles

(E) LRAM, 60 rectangles

46. **Multiple Choice Estimating Derivative Values** Given a continuous function \(f\), which of the following expressions will typically produce the most accurate estimate of \(f'(a)\)? \(E\)

(A) \(\frac{f(a + 0.05) - f(a - 0.05)}{0.05}\)

(B) \(\frac{f(a + 0.05) - f(a - 0.05)}{0.1}\)

(C) \(\frac{f(a + 0.01) - f(a)}{0.01}\)

(D) \(\frac{f(a + 0.01) - f(a - 0.01)}{0.01}\)

(E) \(\frac{f(a + 0.01) - f(a - 0.01)}{0.02}\)

47. **Multiple Choice Using a Numerical Integral** Which of the following cannot be estimated using a numerical integral? \(C\)

(A) The area under a curve that represents some function \(f(x)\)

(B) The distance traveled, when the velocity function is known

(C) The instantaneous velocity of an object, when the position function is known

(D) The change of a city’s population over a 10-year period, when the rate-of-change function is known

(E) The change of a child’s height over a 4-year period, when the rate-of-change function is known

48. **Multiple Choice Using a Numerical Derivative** Which of the following cannot be estimated using a numerical derivative? \(D\)

(A) The instantaneous velocity of an object, when the position function is known

(B) The slope of a curve that represents some function \(g(x)\)

(C) The growth rate of a city’s population, when the population is known as a function of time

(D) The area under a curve that represents some function \(f(x)\)

(E) The rate of change of an airplane’s altitude, when the altitude is known as a function of time

### Explorations

49. Let \(f(x) = 2x^2 + 3x + 1\) and \(g(x) = x^3 + 1\).

(a) Compute the derivative of \(f\). \(4x + 3\)

(b) Compute the derivative of \(g\). \(3x^2\)

(c) Using \(x = 2\) and \(h = 0.001\), compute the standard difference quotient \(\frac{11.002}{h}\)

and the symmetric difference quotient \(\frac{11}{2h}\).

(d) Using \(x = 2\) and \(h = 0.001\), compare the approximations to \(f'(2)\) in part (c). Which is the better approximation?

(e) Repeat parts (c) and (d) for \(g\).

50. **When Are Derivatives and Areas Equal?** Let \(f(x) = 1 + e^x\).

(a) Draw a graph of \(f\) for \(0 \leq x \leq 1\).

(b) Use NDER on your calculator to compute the derivative of \(f\) at \(1. \approx 2.72\)

(c) Use NINT on your calculator to compute the area under \(f\) from \(x = 0\) to \(x = 1\) and compare it to the answer in part (b).

(d) **Group Activity** What do you think the exact answers to parts (b) and (c) are?
51. **Calculator Failure** Many calculators report that NDER of \( f(x) = |x| \) evaluated at \( x = 0 \) is equal to 0. Explain why this is incorrect. Explain why this error occurs.

52. **Grapher Failure** Graph the function \( f(x) = |x|/x \) in the window \([-5, 5] \times [-3, 3]\) and explain why \( f'(0) \) does not exist. Find the value of NDER \( f(0) \) on the calculator and explain why it gives an incorrect answer.

### Extending the Ideas

53. **Group Activity Finding Total Area** The total area bounded by the graph of the function \( y = f(x) \) and the \( x \)-axis from \( x = a \) to \( x = b \) is the area below the graph of \( y = |f(x)| \) from \( x = a \) to \( x = b \).

(a) Find the total area bounded by the graph of \( f(x) = \sin x \) and the \( x \)-axis from \( x = 0 \) to \( x = 2\pi \).

(b) Find the total area bounded by the graph of \( f(x) = x^2 - 2x - 3 \) and the \( x \)-axis from \( x = 0 \) to \( x = 5 \).

54. **Writing to Learn** If a function is unbounded in an interval \([a, b]\) it may have finite area. Use your knowledge of limits at infinity to explain why this might be the case.

55. **Writing to Learn** Let \( f \) and \( g \) be two continuous functions with \( f(x) \geq g(x) \) on an interval \([a, b]\). Devise a limit of sums definition of the area of the region between the two curves. Explain how to compute the area if the area under both curves is already known.

56. **Area as a Function** Consider the function \( f(t) = t^2 \).

(a) Use NINT on a calculator to compute \( A(x) \) where \( A \) is the area under the graph of \( f \) from \( t = 0 \) to \( t = x \) for \( x = 0.25, 0.5, 1, 1.5, 2, 2.5, \) and 3.

(b) Make a table of pairs \( (x, A(x)) \) for the values of \( x \) given in part (a) and plot them using graph paper. Connect the plotted points with a smooth curve.

(c) Use a quadratic regression equation to model the data in part (b) and overlay its graph on a scatter plot of the data.

(d) Make a conjecture about the exact value of \( A(x) \) for any \( x \) greater than zero.

(e) Find the derivative of the \( A(x) \) found in part (d). Record any observations.

57. **Area as a Function** Consider the function \( f(t) = 3t^2 \).

(a) Use NINT on a calculator to compute \( A(x) \) where \( A \) is the area under the graph of \( f \) from \( t = 0 \) to \( t = x \) for \( x = 0.25, 0.5, 1, 1.5, 2, 2.5, \) and 3.

(b) Make a table of pairs \( (x, A(x)) \) for the values of \( x \) given in part (a) and plot them using graph paper. Connect the plotted points with a smooth curve.

(c) Use a cubic regression equation to model the data in part (b) and overlay its graph on a scatter plot of the data.

(d) Make a conjecture about the exact value of \( A(x) \) for any \( x \) greater than zero.

(e) Find the derivative of the \( A(x) \) found in part (d). Record any observations.

58. **Group Activity** Based on Exercises 56 and 57, discuss how derivatives (slope functions) and integrals (area functions) may be connected.
CHAPTER 10 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–4, use the graph of the function \( y = f(x) \) to find (a) \( \lim_{x \to a} f(x) \) and (b) \( \lim_{x \to a} f(x) \).

1. \[
\begin{array}{c}
\text{Graph of } y = f(x) \\
\text{for } x \in [-4, 4]
\end{array}
\]

2. \[
\begin{array}{c}
\text{Graph of } y = f(x) \\
\text{for } x \in [-4, 4]
\end{array}
\]

In Exercises 5–10, find the limit at the indicated point, if it exists. Support your answer graphically.

9. \( f(x) = 2 \tan^{-1} x, \ x = 0 \)

10. \( f(x) = \frac{2}{1 - 2^7}, \ x = 0 \) none

In Exercises 11–14, find the limit. Support your answer with an appropriate table.

11. \( \lim_{x \to -\infty} \frac{-1}{(x + 2)^2} \)

12. \( \lim_{x \to -\infty} \frac{x + 5}{x - 3} \)

13. \( \lim_{x \to \infty} \frac{2 - x^2}{x} \)

14. \( \lim_{x \to -\infty} \frac{x^2}{x - 2} \)

In Exercises 15–18, find the limit.

15. \( \lim_{x \to 2} \frac{1}{x - 2} \)

16. \( \lim_{x \to 2} \frac{1}{x^2 - 4} \)

17. \( \lim_{x \to 0} \frac{1/(2 + x) - 1/2}{x} \)

18. \( \lim_{x \to 0} (2 + x)^3 - 8 \)

In Exercises 19–20, find the vertical and horizontal asymptotes, if any.

19. \( f(x) = \frac{x - 5}{x^2 + 6x + 5} \)

20. \( f(x) = \frac{x^2 + 1}{2x - 4} \)

In Exercises 21–26, find the limit algebraically.

21. \( \lim_{x \to 0} \frac{x^2 + 2x - 15}{3 - x} \)

22. \( \lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} \)

23. \( \lim_{x \to 0} \frac{1/(3 + x) + 1/3}{x} \)

24. \( \lim_{x \to 0} (x - 2) \)

25. \( \lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 3x + 2} \)

26. \( \lim_{x \to 3} (x - 3)^2/x - 3 \)

In Exercises 27 and 28, state a formula for the continuous extension of the function. (See Exercise 90, Section 10.3.)

27. \( f(x) = \frac{x^3 - 1}{x - 1} \)

28. \( f(x) = \frac{x^2 - 6x + 5}{x - 5} \)
In Exercises 29 and 30, use the limit definition to find the derivative of the function at the specified point, if it exists. Support your answer numerically with an NDER calculator estimate.

29. \( f(x) = 1 - x - 2x^2 \) at \( x = 2 \)

30. \( f(x) = (x + 3)^2 \) at \( x = 2 \)

In Exercises 31 and 32, find (a) the average rate of change of the function over the interval \([3, 3.01]\) and (b) the instantaneous rate of change at \( x = 3 \).

31. \( f(x) = x^2 + 2x - 3 \)

32. \( f(x) = \frac{3}{x + 2} \)

In Exercises 33 and 34, find (a) the slope and (b) an equation of the line tangent to the graph of the function at the indicated point.

33. \( f(x) = x^3 - 2x + 1 \) at \( x = 1 \); \( y = x - 1 \)

34. \( f(x) = \sqrt{x - 4} \) at \( x = 7 \)

In Exercises 35 and 36, find the derivative of \( f \).

35. \( f(x) = 5x^2 + 7x - 1 \)

36. \( f(x) = 2 - 8x + 3x^2 \)

In Exercises 37 and 38, complete the following for the indicated interval \([a, b] \).

(a) Verify the given function is nonnegative.

(b) Use a calculator to find the LRAM, RRAM, and average approximations for the area under the graph of the function from \( x = a \) to \( x = b \) with 50 approximating rectangles.

37. \( f(x) = (x - 5)^2 \); \([0, 4]\)

38. \( f(x) = 2x^3 - 3x + 1 \); \([1, 5]\)

39. **Gasoline Prices** The annual average retail price for unleaded regular gasoline in the United States for the years 1990–2003 is given in Table 10.12.

<table>
<thead>
<tr>
<th>Year</th>
<th>Price (cents per gallon)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1990</td>
<td>116.4</td>
</tr>
<tr>
<td>1991</td>
<td>114.0</td>
</tr>
<tr>
<td>1992</td>
<td>112.7</td>
</tr>
<tr>
<td>1993</td>
<td>110.8</td>
</tr>
<tr>
<td>1994</td>
<td>111.2</td>
</tr>
<tr>
<td>1995</td>
<td>114.7</td>
</tr>
<tr>
<td>1996</td>
<td>123.1</td>
</tr>
<tr>
<td>1997</td>
<td>123.4</td>
</tr>
<tr>
<td>1998</td>
<td>105.9</td>
</tr>
<tr>
<td>1999</td>
<td>116.5</td>
</tr>
<tr>
<td>2000</td>
<td>151.0</td>
</tr>
<tr>
<td>2001</td>
<td>146.1</td>
</tr>
<tr>
<td>2002</td>
<td>135.8</td>
</tr>
<tr>
<td>2003</td>
<td>159.1</td>
</tr>
</tbody>
</table>


(a) Draw a scatter plot of the data in Table 10.12. Use \( x = 0 \) for 1990, \( x = 1 \) for 1991, and so forth.

(b) Find the average rate of change from 1990 to 1991 and from 1997 to 1998.

(c) From what year to the next consecutive year does the average rate exhibit the greatest increase?

(d) From what year to the next consecutive year does the average rate exhibit the greatest decrease?

(e) Find a linear regression model for the data and overlay its graph on a scatter plot of the data.

(f) **Group Activity** Find a cubic regression model for the data and overlay its graph on a scatter plot of the data. Discuss pro and con arguments that this is a good model for the gasoline price data. Compare the cubic model with the linear model. Which one does your group think is best? Why?

(g) **Writing to Learn** Use the cubic regression model found in part (f) and NDER to find the instantaneous rate of change in 1997, 1998, 1999, and 2000. How could the cubic model give some misleading information?

(h) **Writing to Learn** Use the cubic regression model found in part (f) to predict the average price of a gallon of unleaded regular gasoline in 2007. Do you think this is a reasonable estimate? Give reasons.

40. **An Interesting Connection**

Let \( A(x) = \text{NINT} \left( \cos t, t, 0, x \right) \)

(a) Draw a scatter plot of the pairs \((x, A(x))\) for \( x = 0, 0.4, 0.8, 0.12, \ldots, 6.0, 6.4 \).

(b) Find a function that seems to model the data in part (a) and overlay its graph on a scatter plot of the data.

(c) Assuming that the function found in part (b) agrees with \( A(x) \) for all values of \( x \), what is the derivative of \( A(x) \)?

(d) **Writing to Learn** Describe what seems to be true about the derivative of \( \text{NINT} \left( f(t), t, 0, x \right) \).
CHAPTER 10 Project

Estimating Population Growth Rates

Las Vegas, Nevada and its surrounding cities represent one of the fastest-growing areas in the United States. Clark County, the county in which Las Vegas is located, has grown by over one million people in the past three decades. The data in the table below (obtained from the web site http://cber.unlv.edu/pop.html) summarizes the growth of Clark County from 1970 through 2004.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>277,230</td>
</tr>
<tr>
<td>1980</td>
<td>463,087</td>
</tr>
<tr>
<td>1990</td>
<td>770,280</td>
</tr>
<tr>
<td>1995</td>
<td>1,055,435</td>
</tr>
<tr>
<td>1999</td>
<td>1,327,145</td>
</tr>
<tr>
<td>2000</td>
<td>1,394,440</td>
</tr>
<tr>
<td>2001</td>
<td>1,485,855</td>
</tr>
<tr>
<td>2002</td>
<td>1,549,657</td>
</tr>
<tr>
<td>2003</td>
<td>1,620,748</td>
</tr>
<tr>
<td>2004</td>
<td>1,715,337</td>
</tr>
</tbody>
</table>

EXPLORATIONS

1. Enter the data in the table above into your graphing calculator or computer. (Let \( t = 0 \) represent 1970.) Make a scatter plot of the data.


3. Use your calculator or computer to find an exponential regression equation to model the population data set (see your grapher’s guidebook for instructions on how to do this).

4. Use the exponential model you just found in question (3) and your calculator/computer NDER feature to estimate the instantaneous population growth rate in 2004. Which of the average growth rates you found in question (2) most closely matches this instantaneous growth rate? Explain why this makes sense.

5. Use the exponential regression model you found in question (3) to predict the population of Clark County in the years 2010, 2020, and 2030. Compare your predictions with the predictions in the table below, obtained from the web site www.co.clark.nv.us. Which predictions seem more reasonable? Explain.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>2,089,102</td>
</tr>
<tr>
<td>2020</td>
<td>2,578,221</td>
</tr>
<tr>
<td>2030</td>
<td>2,941,398</td>
</tr>
</tbody>
</table>

Source: http://www.co.clark.nv.us/comprehensive_planning/Advanced/Demographics/Population_Forecasts/Pop_Forecast_2002to2035.htm.